# Stable Formality Quasi-isomorphisms for Hochschild Cochains I

V. A. Dolgushev

*To my mother*

**Abstract**

We consider $L_\infty$-quasi-isomorphisms for Hochschild cochains whose structure maps admit "graphical expansion". We introduce the notion of stable formality quasi-isomorphism which formalizes such an $L_\infty$-quasi-isomorphism. We define a homotopy equivalence on the set of stable formality quasi-isomorphisms and prove that the set of homotopy classes of stable formality quasi-isomorphisms form a torsor for the group corresponding to the zeroth cohomology of the full (directed) graph complex. This result may be interpreted as a complete description of homotopy classes of formality quasi-isomorphisms for Hochschild cochains in the "stable setting".

## Contents

# 1 Introduction

When a difficult problem is solved, it becomes even more challenging to describe all possible solutions to that problem. In this paper we propose a framework in which this interesting question can be answered completely for Kontsevich's formality conjecture [23] on Hochschild cochain complex.

Kontsevich's formality conjecture [23] states that there exists an $L_\infty$ quasi-isomorphism from the graded Lie algebra $V_A$ of polyvector fields on an affine space to the dg Lie algebra of Hochschild cochains $C^\bullet(A)$ of the algebra of functions $A$ on this affine space.

In plain English the question was to find an infinite collection of maps

$$U_n : \big(V_A\big)^{\otimes n} \to C^{\bullet}(A)\,, \qquad n \ge 1 \tag{1.1}$$

compatible with the action of symmetric groups and satisfying an intricate sequence of relations. The first relation says that $U_1$ is a map of complexes, the second relation says that $U_1$ is compatible with the Lie brackets up to homotopy with $U_2$ serving as a chain homotopy and so on.

In his groundbreaking paper [24] M. Kontsevich proposed a construction of such an $L_\infty$ quasi-isomorphism over reals. His construction is "natural" in the following sense. Given polyvector fields $v_1, v_2, \dots, v_n \in V_A$, the $n$-th component $U_n$ produces a Hochschild cochain via contracting indices of derivatives of various orders of polyvector fields and of functions which enter as arguments for this cochain.

Thus each term in $U_n$ can encoded by a directed graph with two types of vertices: vertices of one type are reserved for polyvector fields and vertices of another type are reserved for functions.

In this paper we formalize the notion of $L_\infty$ quasi-isomorphism for Hochschild cochains which are "natural" in the above sense. In other words, each term in $U_n$ is encoded by a graph with two types of vertices and all the desired identities hold universally, i.e. on the level of linear combinations of graphs.

Such formality quasi-isomorphisms are defined for affine spaces of all[1] (finite) dimensions simultaneously. This is why we refer to them as *stable formality quasi-isomorphisms*. We show that the notion of homotopy equivalence of formality quasi-isomorphisms can also be formulated in this "stable setting". Thus we can talk about homotopy classes of stable formality quasi-isomorphisms.

In this paper we show (see Theorem 6.2) that the set of homotopy classes of stable formality quasi-isomorphisms form a torsor for a pro-unipotent group which is obtained by exponentiating the Lie algebra $H^0(\mathsf{dfGC})$, where $\mathsf{dfGC}$ denotes the full (directed) version of Kontsevich's graph complex [23, Section 5].

Following T. Willwacher [37] the group $\exp(H^0(\mathsf{dfGC}))$ is isomorphic to the Grothendieck-Teichmüller group $\mathsf{GRT}$ introduced by V. Drinfeld in [10].

Thus combining Theorem 6.2 with the result of T. Willwacher [37] we conclude that the set of homotopy classes of stable formality quasi-isomorphisms is a $\mathsf{GRT}$-torsor.

Since a formality quasi-isomorphism for Hochschild cochains provides us with a bijection between equivalence classes of star products and equivalence classes of formal Poisson structures, the result may be interpreted as a complete description of all (deformation) quantization procedures.

To give a precise definition of a stable formality quasi-isomorphism, we recall [21] that an open-closed homotopy algebra ($\mathsf{OC}$-algebra) is a pair $(\mathcal{V}, \mathcal{A})$ of cochain complexes with the following data:

- an $L_\infty$-structure on $\mathcal{V}$,
- an $A_\infty$-structure on $\mathcal{A}$ and
- an $L_\infty$-morphism from $\mathcal{V}$ to the Hochschild cochain complex $C^{\bullet}(\mathcal{A})$ of $\mathcal{A}$.

[1] In fact they are also defined for any $\mathbb{Z}$-graded affine space.

We denote by $\mathsf{OC}$ the 2-colored dg operad which governs open-closed homotopy algebras. It is known that $\mathsf{OC}$ is free as an operad in the category of graded vector spaces. Furthermore, $\mathsf{OC}$ is the cobar construction of a (2-colored) cooperad closely connected with the homology of Voronov's Swiss Cheese operad [36].

Let us also denote by $\mathsf{KGra}$ the 2-colored operad which is "assembled" from graphs used in Kontsevich's paper [24]. This is an operad of graded vector spaces which extends the operad $\mathsf{dGra}$ of directed labeled graphs and acts naturally on the pair "polyvector fields $V_A$ and polynomials $A$". (See Section 3 for more details.)

Let us observe that any map of (dg) operads from $\mathsf{OC}$ to $\mathsf{KGra}$ induces an open-closed homotopy algebra on the pair $(V_A, A)$. So we define a stable formality quasi-isomorphism as a map of (dg) operads from $\mathsf{OC}$ to $\mathsf{KGra}$ subject to a few "boundary conditions". These conditions guarantee that

- the $L_\infty$-structure on polyvector fields coincides with the Lie algebra structure given by the Schouten-Nijenhuis bracket,
- the $A_\infty$-structure on $A$ coincides with the usual associative (and commutative) algebra structure on polynomials, and
- the $L_\infty$-morphism from polyvector fields $V_A$ to Hochschild cochains $C^\bullet(A)$ starts with the Hochschild-Kostant-Rosenberg [19] embedding

$$V_A \hookrightarrow C^\bullet(A)\,.$$

This operadic definition of a stable formality quasi-isomorphism allows us to introduce a natural notion of homotopy equivalence between two stable formality quasi-isomorphisms. We give this definition using an interpretation of stable formality quasi-isomorphisms as Maurer-Cartan elements of an auxiliary dg Lie algebra.

Let us denote by $\mathcal{Z}^0(\mathsf{dfGC})$ the Lie algebra of degree zero cocycles of the full (directed) version $\mathsf{dfGC}$ of Kontsevich's graph complex [23, Section 5]. It is not hard to see that $\mathcal{Z}^0(\mathsf{dfGC})$ is a pro-nilpotent Lie algebra. Hence it can be exponentiated to the group $\exp\big(\mathcal{Z}^0(\mathsf{dfGC})\big)$.

We show that the group $\exp\big(\mathcal{Z}^0(\mathsf{dfGC})\big)$ acts on stable formality quasi-isomorphisms and this action descends to an action of the group $\exp\big(H^0(\mathsf{dfGC})\big)$ on homotopy classes of stable formality quasi-isomorphisms. Finally, we prove that this action of $\exp\big(H^0(\mathsf{dfGC})\big)$ on homotopy classes is simply transitive.

Specialists can probably start reading this paper with Section 3. The goal of Section 2 is mostly to fix conventions and remind a few constructions for colored (co)operads. In Section 3, we define the operad of graded vector spaces $\mathsf{dGra}$ and its 2-colored extension $\mathsf{KGra}$. In this section we also introduce a natural action of $\mathsf{KGra}$ on the pair "polyvector fields $V_A$ and polynomials $A$". In Section 4, we remind the (dg) operad $\mathsf{OC}$ which governs open-closed homotopy algebras [21]. In Section 5, we introduce stable formality quasi-isomorphisms and define a notion of homotopy equivalence between them. Section 6 is devoted to the full graph complex $\mathsf{dfGC}$ and its "action" on stable formality quasi-isomorphisms. In Section 6.1 we also formulate results on $\mathsf{dfGC}$ from [37] which are used in this paper. The main result of this paper (Theorem 6.2) is stated at the end of Section 6. Its proof occupies Section 7 and 8 and it depends on a few technical statements which are proved in Appendices at the end of the paper. Section 9 is devoted to a modification of the notion of stable formality

quasi-isomorphism. This modification is based on graphs without loops (i.e. cycles of length one).

In subsequent paper [7] we will extend Tamarkin's construction [18], [32] to the stable setting and show that this construction gives us an isomorphism from the $\mathsf{GRT}$-torsor of Drinfeld's associators to the $\mathsf{GRT}$-torsor of stable formality quasi-isomorphisms.

**Acknowledgment.** In many respects this work was inspired by papers [23], [24], [25], and [37] and I would like to thank Thomas Willwacher for numerous discussions of his work [37] and for his comments on the original draft of this paper. I would like to thank Chris Rogers for collaboration on lecture notes [9] and for numerous discussions. The results of this paper were presented at the Deformation Theory seminar at Penn. I would like to thank Jim Stasheff for the useful comments he made during my talk. The results of this paper were also presented at XXX Workshop on Geometric Methods in Physics which was held in June 2011, in Bialowieza, Poland. The cost of my trip to this conference was partially covered by the NSF grant # 1124929, which I acknowledge. I am partially supported by the NSF grant DMS 0856196, and the grant FASI RF 14.740.11.0347.

## 2 Preliminaries

We denote by $\mathbb{K}$ a field of characteristic zero. Our underlying symmetric monoidal category $\mathfrak{C}$ is either the category $\mathsf{grVect}_{\mathbb{K}}$ of $\mathbb{Z}$-graded $\mathbb{K}$-vector spaces or the category $\mathsf{Ch}_{\mathbb{K}}$ of unbounded cochain complexes of $\mathbb{K}$-vector spaces. The notation $\mathrm{ad}_\xi$ is reserved for the adjoint action $[\xi,\ ]$ of a vector $\xi$ in a Lie algebra and the expression $\mathrm{CH}(x,y)$ denotes the Campbell-Hausdorff series in variables $x$ and $y$.

The notation $S_n$ is reserved for the group of permutations of the set $\{1,2,\dots,n\}$ and $\mathrm{Sh}_{p_1,p_2,\dots,p_k}$, with $p_i \ge 0$ and $p_1+p_2+\dots+p_k = n$, denotes the subset of $(p_1,p_2,\dots,p_k)$-shuffles in $S_n$, i.e.

$$\mathrm{Sh}_{p_1,p_2,\dots,p_k} = \Big\{ \sigma \in S_n \ \Big|\ \sigma(1) < \dots < \sigma(p_1), \tag{2.1}$$
$$\sigma(p_1+1) < \dots < \sigma(p_1+p_2), \quad \dots, \quad \sigma(n-p_k+1) < \dots < \sigma(n) \Big\}.$$

We often denote by id the identity element of $S_n$ without specifying the number $n$.

We denote by $\mathsf{Com}$ (resp. $\mathsf{As}$) the operad which governs commutative (and associative) algebras without unit (resp. associative algebras without unit). The notation $\mathsf{Lie}$ is reserved for the operad which governs Lie algebras.

Dually, we denote by $\mathsf{coCom}$ (resp. $\mathsf{coAs}$) the cooperad which governs cocommutative (and coassociative) coalgebras without counit (resp. coassociative coalgebras without counit).

The notation $\Lambda$ is reserved for the following collection in $\mathsf{grVect}_{\mathbb{K}}$

$$\Lambda(n) = \begin{cases} \mathbf{s}^{1-n}\mathrm{sgn}_n & \text{if } \ n \ge 1\,, \\ \mathbf{0} & \text{if } \ n = 0\,, \end{cases} \tag{2.2}$$

where $\mathrm{sgn}_n$ is the sign representation of $S_n$.

The collection (2.2) is equipped with a natural structure of an operad and a natural structure of a cooperad. Namely, the $i$-th elementary insertion and the $i$-th elementary co-insertion are given by the formula

$$1_n \circ_i 1_k = (-1)^{(1-k)(n-i)} 1_{n+k-1} \tag{2.3}$$

and the formula

$$\Delta_i(1_{n+k-1}) = (-1)^{(1-k)(n-i)} 1_n \otimes 1_k\,, \tag{2.4}$$

respectively. Here $1_m$ denotes the generator $\mathbf{s}^{1-m}1 \in \mathbf{s}^{1-m}\mathrm{sgn}_m$.

For an operad $\mathcal{O}$ (resp. a cooperad $\mathcal{C}$) we denote by $\Lambda\mathcal{O}$ (resp. $\Lambda\mathcal{C}$) the operad (resp. the cooperad) which is obtained from $\mathcal{O}$ (resp. $\mathcal{C}$) by tensoring with $\Lambda$. For example, a $\Lambda\mathsf{Lie}$-algebra in $\mathsf{grVect}_{\mathbb{K}}$ is a graded vector space $\mathcal{V}$ equipped with the binary operation:

$$\{\,,\,\} : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}$$

of degree $-1$ satisfying the identities:

$$\{v_1, v_2\} = (-1)^{|v_1||v_2|}\{v_2, v_1\}\,,$$

$$\{\{v_1, v_2\}, v_3\} + (-1)^{|v_1|(|v_2|+|v_3|)}\{\{v_2, v_3\}, v_1\} + (-1)^{|v_3|(|v_1|+|v_2|)}\{\{v_3, v_1\}, v_2\} = 0\,,$$

where $v_1, v_2, v_3$ are homogeneous vectors in $\mathcal{V}$.

The operad $\Lambda\mathsf{Lie}$ has the following free resolution

$$\Lambda\mathsf{Lie}_\infty = \mathrm{Cobar}(\Lambda^2\mathsf{coCom}) \tag{2.5}$$

which we use to define an $\infty$-version of $\Lambda\mathsf{Lie}$-algebra structure. Thus a $\Lambda\mathsf{Lie}_\infty$-*structure* on a cochain complex $V$ is a MC element $Q$ in the Lie algebra

$$\mathrm{Coder}\big(\Lambda^2\mathsf{coCom}(V)\big)$$

of coderivations of the cofree coalgebra $\Lambda^2\mathsf{coCom}(V)$ subject to the auxiliary technical condition

$$Q\Big|_V = 0\,.$$

A $\Lambda\mathsf{Lie}_\infty$-*morphism* between $\Lambda\mathsf{Lie}$-algebras $(V, Q)$ and $(W, \widetilde{Q})$ is a homomorphism of the cofree coalgebras

$$\Lambda^2\mathsf{coCom}(V) \qquad \text{and} \qquad \Lambda^2\mathsf{coCom}(W)$$

compatible with the differentials $\partial_V + \mathrm{ad}_Q$ and $\partial_W + \mathrm{ad}_{\widetilde{Q}}$ on $\Lambda^2\mathsf{coCom}(V)$ and $\Lambda^2\mathsf{coCom}(W)$, respectively.

### 2.1 Trees

In this paper all trees are rooted and the root vertex has always valency 1. (Such trees are sometimes called *planted*). The remaining vertices of valency 1 are called *leaves*. A vertex is called *internal* if it is neither a root nor a leaf. We always orient trees in the direction towards the root. Thus every internal vertex has at least 1 incoming edge and exactly 1 outgoing edge. An edge adjacent to a leaf is called *external*.

Unless stated otherwise, we require that a tree has at least one internal vertex.

Let us recall that for every planar tree $\mathbf{t}$ the set of its vertices is equipped with a natural total order. To define this total order on the set $V(\mathbf{t})$ of all vertices of $\mathbf{t}$ we introduce the function

$$\mathcal{N} : V(\mathbf{t}) \to V(\mathbf{t})\,. \tag{2.6}$$

To a non-root vertex $v$ the function $\mathcal{N}$ assigns the next vertex along the (unique) path connecting $v$ to the root vertex. Furthermore $\mathcal{N}$ sends the root vertex to the root vertex.

Let $v_1, v_2$ be two distinct vertices of $\mathbf{t}$. If $v_1$ lies on the path which connects $v_2$ to the root vertex then we declare that

$$v_1 < v_2 .$$

Similarly, if $v_2$ lies on the path which connects $v_1$ to the root vertex then we declare that

$$v_2 < v_1 .$$

If neither of the above options realize then there exist numbers $k_1$ and $k_2$ such that

$$\mathcal{N}^{k_1}(v_1) = \mathcal{N}^{k_2}(v_2) \tag{2.7}$$

but

$$\mathcal{N}^{k_1-1}(v_1) \neq \mathcal{N}^{k_2-1}(v_2) .$$

Since the tree $\mathbf{t}$ is planar the set of $\mathcal{N}^{-1}(\mathcal{N}^{k_1}(v_1))$ is equipped with a total order. Furthermore, since both vertices $\mathcal{N}^{k_1-1}(v_1)$ and $\mathcal{N}^{k_2-1}(v_2)$ belong to the set $\mathcal{N}^{-1}(\mathcal{N}^{k_1}(v_1))$, we may compare them with respect to this order.

We declare that, if $\mathcal{N}^{k_1-1}(v_1) < \mathcal{N}^{k_2-1}(v_2)$, then

$$v_1 < v_2 .$$

Otherwise we set $v_2 < v_1$.

It is not hard to see that the resulting relation $<$ on $V(\mathbf{t})$ is indeed a total order.

We often restrict this total order on $V(\mathbf{t})$ to the subset $V_{int}(\mathbf{t})$ of internal vertices of $\mathbf{t}$. This way we always think of $V_{int}(\mathbf{t})$ as a totally ordered set. Keeping this order in mind, we often say things like "the first internal vertex", "the second internal vertex", and "the $i$-th internal vertex".

We have an obvious bijection between the set of edges $E(\mathbf{t})$ of a tree $\mathbf{t}$ and the subset of vertices:

$$V(\mathbf{t}) \setminus \{\text{root vertex}\} . \tag{2.8}$$

This bijection assigns to a vertex $v$ in (2.8) its outgoing edge.

Thus the canonical total order on the set (2.8) gives us a natural total order on the set of edges $E(\mathbf{t})$.

For our purposes we also extend the total orders on the sets $V(\mathbf{t}) \setminus \{\text{root vertex}\}$ and $E(\mathbf{t})$ to the disjoint union

$$\Big( V(\mathbf{t}) \setminus \{\text{root vertex}\} \Big) \sqcup E(\mathbf{t}) \tag{2.9}$$

by declaring that a vertex is bigger than its outgoing edge. For example, the root edge is the minimal element in the set (2.9).

#### 2.1.1 Height function

For every tree $\mathbf{t}$ we have the obvious function

$$\mathrm{ht} : V(\mathbf{t}) \to \{0, 1, 2, 3 \dots\} \tag{2.10}$$

from the set of vertices to the set of non-negative integers. The function assigns to a vertex $v$ the length of the path from this vertex to the root vertex. We call $\mathrm{ht}(v)$ the *height* of a vertex $v$.

### 2.1.2 Colored trees, labeled colored trees

Let $\Xi$ be a non-empty finite totally ordered set. We will call elements of $\Xi$ colors.

Let $\mathbf{t}$ be a tree and $v$ be an internal vertex of $\mathbf{t}$. Let us denote by $E_v(\mathbf{t})$ the set of edges terminating at $v$. Recall that a planar structure on a tree $\mathbf{t}$ is nothing but a choice of total orders on the sets $E_v(\mathbf{t})$ for all internal vertices $v$.

A $\Xi$-*colored planar tree* is a planar tree $\mathbf{t}$ equipped with a map

$$c_{\mathbf{t}} : E(\mathbf{t}) \to \Xi$$

which satisfies the following condition

**Condition 2.1** *The restriction of the map $c_{\mathbf{t}}$ to the subset $E_v(\mathbf{t}) \subset E(\mathbf{t})$*

$$c_{\mathbf{t}}\Big|_{E_v(\mathbf{t})} \; : \; E_v(\mathbf{t}) \to \Xi$$

*is a monotonous function for every internal vertex $v$.*

We refer to the value $c_{\mathbf{t}}(e)$ of $c_{\mathbf{t}}$ at $e$ as *the color of the edge $e$*.

Using the obvious bijection between the leaves and the external edges we assign to each leaf the color of its adjacent edge. We denote the resulting color function by $c_{\mathbf{t},l}$

$$c_{\mathbf{t},l} : L(\mathbf{t}) \to \Xi \,, \tag{2.11}$$

where $L(\mathbf{t})$ is the set of leaves of $\mathbf{t}$.

Using the function (2.11) we split the set $L(\mathbf{t})$ into the disjoint union

$$L(\mathbf{t}) = \bigsqcup_{\chi \in \Xi} c^{-1}_{\mathbf{t},l}(\chi) \,. \tag{2.12}$$

Then we define a *labeled $\Xi$-colored planar tree* as a $\Xi$-colored planar tree $\mathbf{t}$ equipped with (not necessarily monotonous) bijections

$$\mathfrak{l}_{\chi} : c^{-1}_{\mathbf{t},l}(\chi) \to \{1, 2, \dots, |c^{-1}_{\mathbf{t},l}(\chi)|\} \,. \tag{2.13}$$

**Example 2.1** In this paper the set $\Xi$ is often the two-element set[2] $\{\mathfrak{c}, \mathfrak{o}\}$ with $\mathfrak{c} < \mathfrak{o}$. Figure 2.1 gives us an example of a labeled $\{\mathfrak{c}, \mathfrak{o}\}$-colored (or simply 2-colored) planar tree. Throughout this paper edges of color $\mathfrak{c}$ are drawn solid and edges of color $\mathfrak{o}$ are drawn dashed.

$\Xi$-colored planar corollas will play an important role. In particular, we will need a map which assigns a $\Xi$-colored planar corolla $\kappa(\mathbf{t})$ to a $\Xi$-colored planar tree $\mathbf{t}$. To define this map we observe that $\Xi$-colored planar corollas are in bijection with the arrays $\{n_\chi; \chi_{root}\}_{\chi \in \Xi}$ where $n_\chi$ are non-negative integers with at least one $n_\chi \neq 0$ and $\chi_{root}$ is an element in $\Xi$. More precisely, the array $\{n_\chi; \chi_{root}\}_{\chi \in \Xi}$ corresponding to a $\Xi$-colored planar corolla $\mathbf{q}$ has $\chi_{root}$ equal to the color of the root edge of $\mathbf{q}$ and

$$n_\chi = |c^{-1}_{\mathbf{q},l}(\chi)| \,. \tag{2.14}$$

For example, the 2-colored corolla depicted on figure 2.2 corresponds to the array $\{2, 1; \mathfrak{o}\}$.

[2] The notation for colors comes from string theory [39]. $\mathfrak{o}$ refers to open strings and $\mathfrak{c}$ refers to closed strings.

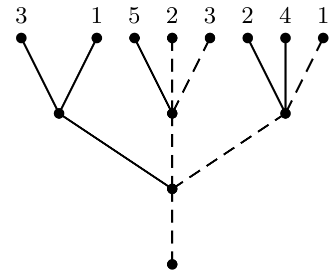



Fig. 2.1: Solid edges carry the color $\mathfrak{c}$ and dashed edges carry the color $\mathfrak{o}$

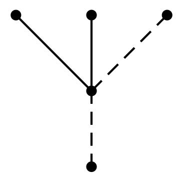


Fig. 2.2: The corolla corresponding to the array $\{2, 1; \mathfrak{o}\}$

We now notice that every $\Xi$-colored planar tree $\mathbf{t}$ gives us the array $\{n_\chi; \chi_{root}\}_{\chi \in \Xi}$ with $\chi_{root}$ being the color of the root edge of $\mathbf{t}$ and

$$n_\chi = |c^{-1}_{\mathbf{t},l}(\chi)|\,.$$

We denote by $\kappa(\mathbf{t})$ the $\Xi$-colored planar corolla corresponding to this array.

For example, the corolla[3] $\kappa(\mathbf{t})$ corresponding to the 2-colored planar tree $\mathbf{t}$ on figure 2.1 is shown on figure 2.3

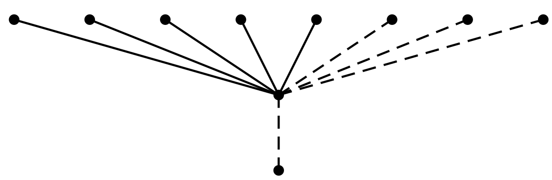


Fig. 2.3: The corolla $\kappa(\mathbf{t})$.

There is also an obvious way to assign a $\Xi$-colored planar corolla to an internal vertex $x$ of a $\Xi$-colored planar tree $\mathbf{t}$. This corolla is formed by the edges adjacent to the vertex $x$ and we denote this corolla by $\kappa(x)$.

**Remark 2.2** It is clear that, if $\Xi$ is a one-point set, then $\Xi$-colored planar trees are exactly non-colored planar trees and $\Xi$-colored corollas are in bijection with positive integers.

### 2.1.3 Groupoid of labeled (colored) planar trees

For our purposes we need to upgrade the set of labeled $\Xi$-colored planar trees to a groupoid $\mathsf{Tree}^{\Xi}$. Objects of $\mathsf{Tree}^{\Xi}$ are labeled $\Xi$-colored planar trees and morphisms are non-planar isomorphisms of the corresponding trees compatible with labeling and coloring in the following sense: an isomorphism $\phi$ from $\mathbf{t}$ to $\mathbf{t}'$ sends the leaf of $\mathbf{t}$ with label $i$ to the leaf of

[3] Note that if the $\Xi$-colored planar tree $\mathbf{t}$ is also labeled then we, first, forget labeling and then form the $\Xi$-colored corolla $\kappa(\mathbf{t})$.

$\mathbf{t}'$ with label $i$; furthermore, if the edge originating at $v \in V(\mathbf{t})$ carries the color $\chi$ then the edge originating at $\phi(v) \in V(\mathbf{t}')$ carries the same color $\chi$.

**Example 2.3** Let us denote by $\mathbf{t}$ the labeled 2-colored planar tree depicted on figure 2.1. The tree $\mathbf{t}_1$ on figure 2.4 is isomorphic to $\mathbf{t}$ while the tree $\mathbf{t}_2$ on figure 2.5 is not isomorphic to $\mathbf{t}$.

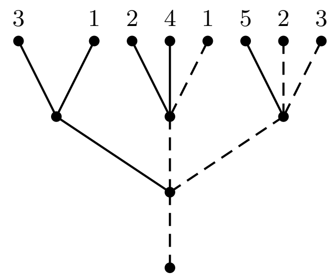



Fig. 2.4: The labeled 2-colored tree $\mathbf{t}_1$

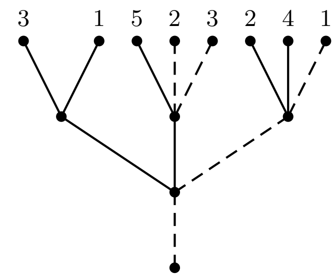



Fig. 2.5: The labeled 2-colored tree $\mathbf{t}_2$

It is easy to see that the automorphism group of every labeled $\Xi$-colored planar tree $\mathbf{t}$ is trivial. In other words, a set of morphisms between any two objects of $\mathsf{Tree}^{\Xi}$ has at most one element. Furthermore, if the corollas $\kappa(\mathbf{t})$ and $\kappa(\mathbf{t}')$ corresponding to labeled $\Xi$-colored planar trees $\mathbf{t}$ and $\mathbf{t}'$ are different then there are no morphisms between $\mathbf{t}$ and $\mathbf{t}'$.

Thus the groupoid $\mathsf{Tree}^{\Xi}$ splits into the disjoint union

$$\mathsf{Tree}^{\Xi} = \bigsqcup_{\mathbf{q}} \mathsf{Tree}^{\Xi}(\mathbf{q}) \tag{2.15}$$

where $\mathsf{Tree}^{\Xi}(\mathbf{q})$ is the full subcategory of labeled $\Xi$-colored planar trees $\mathbf{t}$ satisfying the condition

$$\kappa(\mathbf{t}) = \mathbf{q} \tag{2.16}$$

and the union (2.15) is taken over all $\Xi$-colored planar corollas.

For every $\Xi$-colored planar corolla $\mathbf{q}$ we introduce the group

$$S_{\mathbf{q}} = \prod_{\chi \in \Xi} S_{n_\chi}\,, \tag{2.17}$$

where $n_\chi = c^{-1}_{\mathbf{q},l}(\chi)$. This group acts in the obvious way on the groupoid $\mathsf{Tree}^{\Xi}(\mathbf{q})$ by permuting labels of leaves with the same colors.

We reserve the notation $\mathsf{Tree}^{\Xi}_2(\mathbf{q})$ for the full subcategory of $\mathsf{Tree}^{\Xi}(\mathbf{q})$ whose objects are labeled $\Xi$-colored planar trees with exactly two internal vertices. For example, if $\Xi = \{\mathfrak{c} < \mathfrak{o}\}$ and $\mathbf{q}$ is the corolla corresponding the array $(n, k; \chi)$ then the set of isomorphism classes of objects in $\mathsf{Tree}^{\Xi}_2(\mathbf{q})$ is in bijection with the set

$$\bigsqcup_{1 \le p \le n} \bigsqcup_{1 \le q \le k} \big\{ (\sigma, \tau, \chi_1) \mid \sigma \in \mathrm{Sh}_{p,n-p},\ \tau \in \mathrm{Sh}_{q,k-q},\ \chi_1 \in \{\mathfrak{c}, \mathfrak{o}\} \big\}\,. \tag{2.18}$$

Namely, if the root edge of the corolla $\mathbf{q}$ carries the color $\mathfrak{c}$ then the bijection assigns to an element $(\sigma, \tau, \mathfrak{c})$ (resp. $(\sigma, \tau, \mathfrak{o})$) the isomorphism class of the labeled 2-colored planar tree depicted on figure 2.6 (resp. 2.7). If the root edge of the corolla $\mathbf{q}$ carries the color $\mathfrak{o}$ then

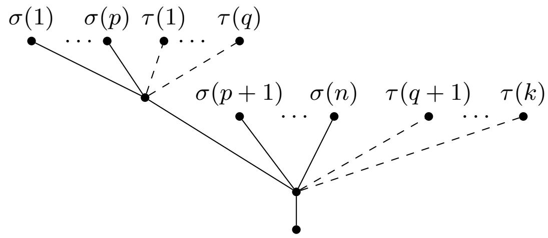



Fig. 2.6: Here $\sigma \in \mathrm{Sh}_{p,n-p}$ and $\tau \in \mathrm{Sh}_{q,k-q}$

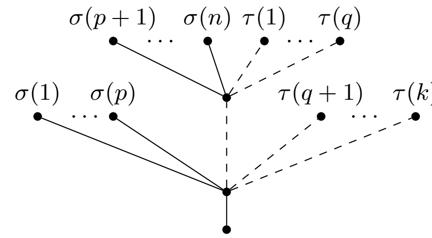



Fig. 2.7: Here $\sigma \in \mathrm{Sh}_{p,n-p}$ and $\tau \in \mathrm{Sh}_{q,k-q}$

we need to replace the solid root edges of the trees depicted on figures 2.6 and 2.7 by dashed edges.

As we mentioned above, the case when $\Xi$ is the one-point set corresponds to non-colored labeled planar trees. Corollas can be identified with positive integers and the groupoid $\mathsf{Tree}$ of labeled planar trees splits into the disjoint union

$$\mathsf{Tree} = \bigsqcup_{n \ge 1} \mathsf{Tree}(n)\,, \tag{2.19}$$

where $\mathsf{Tree}(n)$ is the groupoid of labeled planar trees with exactly $n$ leaves. We refer to objects of $\mathsf{Tree}(n)$ as *$n$-labeled planar trees.*

By analogy with $\mathsf{Tree}_2^{\Xi}(\mathbf{q})$, we reserve the notation $\mathsf{Tree}_2(n)$ for the full sub-groupoid of $\mathsf{Tree}(n)$ whose objects are $n$-labeled planar trees with exactly 2 internal vertices. It is not hard to see that isomorphism classes of $\mathsf{Tree}_2(n)$ are in bijection with the union

$$\bigsqcup_{1 \le p \le n} \mathrm{Sh}_{p,n-p}$$

where $\mathrm{Sh}_{p,n-p}$ denotes the set of $(p, n-p)$-shuffles in $S_n$ . The bijection assigns to a $(p, n-p)$-shuffles $\tau$ the isomorphism class of the planar tree depicted on figure 2.8.

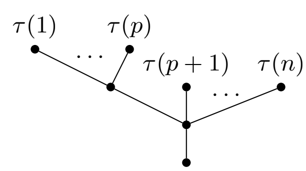



Fig. 2.8: Here $\tau$ is a $(p, n-p)$-shuffle

### 2.1.4 Insertion of (colored) trees

Let $\widetilde{\mathbf{t}}$ be a $\Xi$-colored labeled planar tree and let $x_i$ be its $i$-th internal vertex. Then for every $\Xi$-colored labeled planar tree $\mathbf{t}$ satisfying the condition

$$\kappa(\mathbf{t}) = \kappa(x_i) \tag{2.20}$$

we can define the insertion $\bullet_i$ of the tree $\mathbf{t}$ into the $i$-th internal vertex of $\widetilde{\mathbf{t}}$ . For the resulting planar tree $\widetilde{\mathbf{t}} \bullet_i \mathbf{t}$ we have

$$\kappa(\widetilde{\mathbf{t}} \bullet_i \mathbf{t}) = \kappa(\widetilde{\mathbf{t}})\,. \tag{2.21}$$

To build the tree $\widetilde{\mathbf{t}} \bullet_i \mathbf{t}$, we follow these steps:

- first, we denote by $E_{i,\chi}(\widetilde{\mathbf{t}})$ the set of edges of color $\chi$ terminating at the $i$-th internal vertex of $\widetilde{\mathbf{t}}$. Since $\widetilde{\mathbf{t}}$ is planar, the set $E_{i,\chi}(\widetilde{\mathbf{t}})$ comes with a total order;
- second, we erase the $i$-th internal vertex of $\widetilde{\mathbf{t}}$;
- third, we identify the root edge of $\mathbf{t}$ with the edge of $\widetilde{\mathbf{t}}$ which originated at the $i$-th internal vertex;
- finally, we identify external edges of $\mathbf{t}$ with edges in the union

$$\bigsqcup_{\chi\in\Xi} E_{i,\chi}(\widetilde{\mathbf{t}})$$

following this rule: the external edge with color $\chi$ and label $j$ gets identified with the $j$-th edge in the set $E_{i,\chi}(\widetilde{\mathbf{t}})$. In doing this, we keep the same planar structure on $\mathbf{t}$, so, in general, branches of $\widetilde{\mathbf{t}}$ move around.

**Example 2.4** Figure 2.11 shows the result of the insertion $\widetilde{\mathbf{t}}\bullet_1\mathbf{t}$ of the labeled 2-colored planar tree $\mathbf{t}$ (depicted on figure 2.10) into the first internal vertex of the labeled 2-colored planar tree $\widetilde{\mathbf{t}}$ (depicted on figure 2.9).

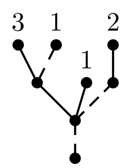


Fig. 2.9: A labeled 2-colored planar tree $\widetilde{\mathbf{t}}$

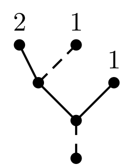


Fig. 2.10: A labeled 2-colored planar tree $\mathbf{t}$

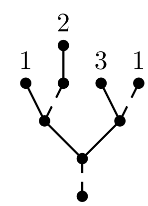


Fig. 2.11: The result of the insertion $\widetilde{\mathbf{t}}\bullet_1\mathbf{t}$

## 2.2 Colored operads and their dual versions

### 2.2.1 Colored collections

Let us recall that a $\Xi$-colored collection in a symmetric monoidal category $\mathfrak{C}$ is given by the data:

— For each $\Xi$-colored planar corolla $\mathbf{q}$ we have an object

$$P(\mathbf{q})\in\mathfrak{C}$$

equipped with a left action of the group $S_{\mathbf{q}}$ (2.17).

Morphisms of $\Xi$-colored collections are defined in the obvious way.

In the case $\Xi=\{\mathfrak{c}<\mathfrak{o}\}$ we will denote the object corresponding to a corolla $\mathbf{q}$ by

$$P(n,k)^{\chi},$$

where $n=|c^{-1}_{\mathbf{q},l}(\mathfrak{c})|$, $k=|c^{-1}_{\mathbf{q},l}(\mathfrak{o})|$, and $\chi$ is the color of the root edge.

Given a $\Xi$-colored collection $P$ in $\mathfrak{C}$ we introduce a covariant functor

$$\underline{P}:\mathsf{Tree}^{\Xi}\to\mathfrak{C} \tag{2.22}$$

from the groupoid $\mathsf{Tree}^{\Xi}$ of $\Xi$-colored labeled planar trees to $\mathfrak{C}$.

To a labeled $\Xi$-colored planar tree $\mathbf{t}$, the functor $\underline{P}$ assigns the object

$$\underline{P}(\mathbf{t}) = \bigotimes_{x \in V_{int}(\mathbf{t})} P(\kappa(x)) \tag{2.23}$$

where $V_{int}(\mathbf{t})$ is the set of all internal vertices of $\mathbf{t}$, $\kappa(x)$ is the $\Xi$-colored planar corolla formed by all edged of $\mathbf{t}$ adjacent to $x$, and the order of the factors agrees with the total order on the set $V_{int}(\mathbf{t})$.

To define $\underline{P}$ on the level of morphisms we use the action of the group (2.17) on $P(\mathbf{q})$ and the braiding of the symmetric monoidal category in the obvious way. For example, let $\mathbf{t}$ and $\mathbf{t}_1$ be 2-colored trees depicted on figures 2.1 and 2.4, respectively. For these trees we have

$$\underline{P}(\mathbf{t}) = P(1,2)^{\mathfrak{o}} \otimes P(2,0)^{\mathfrak{c}} \otimes P(1,2)^{\mathfrak{o}} \otimes P(2,1)^{\mathfrak{o}}\,,$$

$$\underline{P}(\mathbf{t}_1) = P(1,2)^{\mathfrak{o}} \otimes P(2,0)^{\mathfrak{c}} \otimes P(2,1)^{\mathfrak{o}} \otimes P(1,2)^{\mathfrak{o}}\,.$$

The functor $\underline{P}$ sends the unique morphism $\phi : \mathbf{t} \to \mathbf{t}_1$ to

$$\underline{P}(\phi) = (\mathrm{id}, \sigma_{12}) \otimes 1 \otimes \beta,$$

where $(\mathrm{id}, \sigma_{12})$ is the non-identity element of the group $S_1 \times S_2$ and $\beta$ is the braiding

$$\beta : P(1,2)^{\mathfrak{o}} \otimes P(2,1)^{\mathfrak{o}} \to P(2,1)^{\mathfrak{o}} \otimes P(1,2)^{\mathfrak{o}}\,.$$

### 2.2.2 Colored (pseudo)operads

Let $\mathbf{q}$ be a $\Xi$-colored planar corolla. We say that the corolla $\mathbf{q}$ is *naturally labeled* if the bijection

$$\mathfrak{l}_{\chi} : c^{-1}_{\mathbf{t},l}(\chi) \to \{1, 2, \dots |c^{-1}_{\mathbf{t},l}(\chi)|\}$$

is monotonous for every $\chi \in \Xi$. An example of a naturally labeled corolla is depicted on figure 2.12.

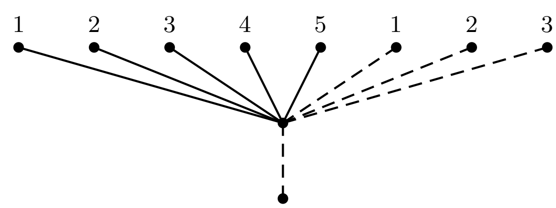



Fig. 2.12: An example of a naturally labeled corolla

For our purposes it is convenient to use the following definition of a colored pseudo-operad.

**Definition 2.1** *A $\Xi$-colored pseudo-operad is a $\Xi$-colored collection $P$ equipped with multiplication maps*

$$\mu_{\mathbf{t}} : \underline{P}(\mathbf{t}) \to P(\kappa(\mathbf{t})) \tag{2.24}$$

*defined for every labeled $\Xi$-colored planar trees $\mathbf{t}$ and subject to the following axioms:*

- *If $\mathbf{q}$ is a naturally labeled $\Xi$-colored planar corolla then*

$$\mu_{\mathbf{q}} = \mathrm{id}_{P(\mathbf{q})}\,. \tag{2.25}$$

- *The operation $\mu_{\mathbf{t}}$ is $S_{\kappa(\mathbf{t})}$-equivariant. Namely, for every labeled $\Xi$-colored planar tree $\mathbf{t}$ we have*
$$\mu_{\sigma(\mathbf{t})} = \sigma \circ \mu_{\mathbf{t}}\,, \qquad \forall \quad \sigma \in S_{\kappa(\mathbf{t})}\,. \tag{2.26}$$
- *For every morphism $\lambda : \mathbf{t} \to \mathbf{t}'$ in $\mathsf{Tree}^{\Xi}$ we have*
$$\mu_{\mathbf{t}'} \circ \underline{P}(\lambda) = \mu_{\mathbf{t}}\,. \tag{2.27}$$
- *To formulate the associativity axiom we consider a triple $(\widetilde{\mathbf{t}}, x, \mathbf{t})$ where $\widetilde{\mathbf{t}}$ is a labeled $\Xi$-colored planar tree, $x$ is the $i$-th internal vertex of $\widetilde{\mathbf{t}}$, and $\mathbf{t}$ is a labeled $\Xi$-colored planar tree such that $\kappa(\mathbf{t}) = \kappa(x)$. The associativity axiom states that for each such triple we have*
$$\mu_{\widetilde{\mathbf{t}}} \circ (1 \otimes \dots \otimes 1 \otimes \underbrace{\mu_{\mathbf{t}}}_{i\text{-th spot}} \otimes 1 \otimes \dots \otimes 1) \circ \beta_{\widetilde{\mathbf{t}},x,\mathbf{t}} = \mu_{\widetilde{\mathbf{t}} \bullet_i \mathbf{t}} \tag{2.28}$$
*where $\widetilde{\mathbf{t}} \bullet_i \mathbf{t}$ is the tree obtained by inserting $\mathbf{t}$ into the $i$-th vertex of $\widetilde{\mathbf{t}}$ and $\beta_{\widetilde{\mathbf{t}},x,\mathbf{t}}$ is the isomorphism in $\mathfrak{C}$ which is "responsible for putting tensor factors in the correct order".*

*Morphisms of pseudo-operads are defined in the obvious way.*

Let $\mathbf{q}_1$ and $\mathbf{q}_2$ be two naturally labeled $\Xi$-colored planar corollas such that the root edge of $\mathbf{q}_2$ carries the color $\chi$. We denote by $\mathbf{t}_{i,\chi}$ the labeled $\Xi$-colored planar tree which is obtained from $\mathbf{q}_1$ and $\mathbf{q}_2$ in two steps. First, we glue $\mathbf{q}_2$ with $\mathbf{q}_1$ by identifying the root edge of $\mathbf{q}_2$ with the external edge of $\mathbf{q}_1$ which carries the color $\chi$ and label $i$. Second, we label the leaves of the resulting $\Xi$-colored planar tree in such a way that for every $\chi' \in \Xi$ the function
$$\mathfrak{l}_{\chi'} : c_{\mathbf{t}_{i,\chi},l} \to \{1, 2, \dots, |c^{-1}_{\mathbf{t}_{i,\chi},l}(\chi')|\}$$
is monotonous. For example, if $\mathbf{q}_1$ and $\mathbf{q}_2$ is the 2-colored corollas depicted on figures 2.13 and 2.14, respectively, then $\mathbf{t}_{2,\mathfrak{o}}$ is the tree depicted on figure 2.15. Although the tree $\mathbf{t}_{i,\chi}$

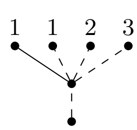


Fig. 2.13: A 2-colored corola $\mathbf{q}_1$

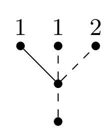


Fig. 2.14: A 2-colored corola $\mathbf{q}_2$

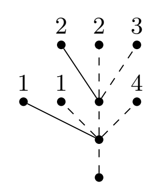


Fig. 2.15: The labeled 2-colored tree $\mathbf{t}_{2,\mathfrak{o}}$

depends on the corollas $\mathbf{q}_1$ and $\mathbf{q}_2$, we suppress $\mathbf{q}_1$ and $\mathbf{q}_2$ from the notation.

To introduce a structure of a pseudo-operad on a collection $P$ it suffices to specify the multiplications
$$\mu_{\mathbf{t}_{i,\chi}} : P(\mathbf{q}_1) \otimes P(\mathbf{q}_2) \to P(\kappa(\mathbf{t}_{i,\chi})) \tag{2.29}$$
for all tuples $(\mathbf{q}_1, \mathbf{q}_2, i, \chi)$. All the remaining multiplications (2.24) can be deduced from (2.29) using axioms of pseudo-operad.

The operations (2.29) are called *elementary insertions* and we will use for them the special notation $\circ_{i,\chi}$. Namely, if $v \in P(\mathbf{q}_1)$ and $w \in P(\mathbf{q}_2)$ then
$$v \circ_{i,\chi} w \,:=\, \mu_{\mathbf{t}_{i,\chi}}(v, w)\,. \tag{2.30}$$

Let $\chi \in \Xi$ and let $\mathbf{u}_\chi$ be the labeled tree with exactly two edges: the root edge and the external edge, both carrying the color $\chi$:

$$\mathbf{u}_\chi = \qquad (2.31)$$



We say that

**Definition 2.2** *$P$ is a $\Xi$-colored operad if $P$ is a $\Xi$-colored pseudo-operad with chosen maps (unit maps)*

$$I_\chi : \mathbb{K} \to P(\mathbf{u}_\chi) \qquad (2.32)$$

*such that the compositions*

$$\begin{array}{l} P(\mathbf{q}) \cong P(\mathbf{q}) \otimes \mathbb{K} \stackrel{1\otimes I_\chi}{\longrightarrow} P(\mathbf{q}) \otimes P(\mathbf{u}_\chi) \stackrel{\mu_{\mathbf{t}_{i,\chi}}}{\longrightarrow} P(\mathbf{q}) \\ P(\mathbf{q}) \cong \mathbb{K} \otimes P(\mathbf{q}) \stackrel{I_\chi\otimes 1}{\longrightarrow} P(\mathbf{u}_\chi) \otimes P(\mathbf{q}) \stackrel{\mu_{\mathbf{t}_{i,\chi}}}{\longrightarrow} P(\mathbf{q}) \end{array} \qquad (2.33)$$

*coincide with the identity map on $P(\mathbf{q})$ whenever they make sense. Morphisms of $\Xi$-colored operads are defined in the obvious way.*

**Remark 2.5** For a conventional definition of colored operads we refer the reader to paper [2] by C. Berger and I. Moerdijk. Due to the observation made in [2, Remark 1.3] the definition given here is equivalent to the conventional one.

**Example 2.6** Let $\Xi = \{\mathfrak{c}, \mathfrak{o}\}$ and $(\mathcal{V}, \mathcal{A})$ be a pair of cochain complexes. The 2-colored collection $\mathsf{End}_{\mathcal{V},\mathcal{A}}$ with

$$\mathsf{End}_{\mathcal{V},\mathcal{A}}(n,k)^{\mathfrak{c}} = \mathrm{Hom}(\mathcal{V}^{\otimes n} \otimes \mathcal{A}^{\otimes k}, \mathcal{V})\,, \qquad \mathsf{End}_{\mathcal{V},\mathcal{A}}(n,k)^{\mathfrak{o}} = \mathrm{Hom}(\mathcal{V}^{\otimes n} \otimes \mathcal{A}^{\otimes k}, \mathcal{A}) \qquad (2.34)$$

is equipped with the obvious structure of a 2-colored operad. $\mathsf{End}_{\mathcal{V},\mathcal{A}}$ is called the endomorphism operad of the pair $(\mathcal{V}, \mathcal{A})$. This example can be obviously generalized to an arbitrary set of colors $\Xi$.

Example 2.6 plays an important role because an algebra over a $\Xi$-colored operad $P$ is defined as a family $\{V_\chi\}_{\chi\in\Xi}$ of objects in $\mathfrak{C}$ with an operad morphism from $P$ to $\mathsf{End}_{\{V_\chi\}_{\chi\in\Xi}}$.

#### 2.2.3 Augmentation of colored operads

The $\Xi$-colored collection

$$*(\mathbf{q}) = \begin{cases} \mathbb{K} & \text{if} \quad \mathbf{q} = \mathbf{u}_\chi \text{ for some } \chi \in \Xi\,, \\ \mathbf{0} & \qquad \text{otherwise} \end{cases} \qquad (2.35)$$

is equipped with a unique structure of a $\Xi$-colored operad. It is easy to see that $*$ is the initial object in the category of $\Xi$-colored operads.

A $\Xi$-colored operad $P$ is called augmented if $P$ comes with an operad morphism

$$\varepsilon : P \to *.$$

For every augmented operad $P$ the kernel of the map $P \to *$ is naturally a pseudo-operad. We denote this pseudo-operad by $P_\circ$.

It is not hard to see that the assignment

$$P \rightsquigarrow P_\circ$$

extends to a functor. According to[4] [27, Proposition 21] this functor gives us an equivalence between the category of augmented (colored) operads and the category of (colored) pseudo-operads.

#### 2.2.4 Colored (pseudo)cooperads

Reversing all arrows in Definition 2.1 we get

**Definition 2.3** *A $\Xi$-colored pseudo-cooperad is a $\Xi$-colored collection $Q$ equipped with comultiplication maps*

$$\Delta_{\mathbf{t}} : Q(\kappa(\mathbf{t})) \to \underline{Q}(\mathbf{t}) \tag{2.36}$$

*defined for every labeled $\Xi$-colored planar trees $\mathbf{t}$ and subject to the following axioms:*

- *If $\mathbf{q}$ is a naturally labeled $\Xi$-colored planar corolla then*

$$\Delta_{\mathbf{q}} = \mathrm{id}_{Q(\mathbf{q})} \,. \tag{2.37}$$

- *The operation $\Delta_{\mathbf{t}}$ is $S_{\kappa(\mathbf{t})}$-equivariant. Namely, for every labeled $\Xi$-colored planar tree $\mathbf{t}$ we have*

$$\Delta_{\sigma(\mathbf{t})} \circ \sigma = \Delta_{\mathbf{t}} \,, \qquad \forall \quad \sigma \in S_{\kappa(\mathbf{t})} \,. \tag{2.38}$$

- *For every morphism $\lambda : \mathbf{t} \to \mathbf{t}'$ in $\mathsf{Tree}^{\Xi}$ we have*

$$\Delta_{\mathbf{t}'} = \underline{Q}(\lambda) \circ \Delta_{\mathbf{t}} \,. \tag{2.39}$$

- *To formulate the coassociativity axiom we consider a triple $(\widetilde{\mathbf{t}}, x, \mathbf{t})$ where $\widetilde{\mathbf{t}}$ is a labeled $\Xi$-colored planar tree, $x$ is the $i$-th internal vertex of $\widetilde{\mathbf{t}}$, and $\mathbf{t}$ is a labeled $\Xi$-colored planar tree such that $\kappa(\mathbf{t}) = \kappa(x)$. The coassociativity axiom states that for each such triple we have*

$$(1 \otimes \cdots \otimes 1 \otimes \underbrace{\Delta_{\mathbf{t}}}_{i\text{-th spot}} \otimes 1 \otimes \cdots \otimes 1)\Delta_{\widetilde{\mathbf{t}}} = \Delta_{\widetilde{\mathbf{t}} \bullet_i \mathbf{t}} \circ \beta_{\widetilde{\mathbf{t}}, x, \mathbf{t}} \tag{2.40}$$

  *where $\widetilde{\mathbf{t}} \bullet_i \mathbf{t}$ is the tree obtained by inserting $\mathbf{t}$ into the $i$-th vertex of $\widetilde{\mathbf{t}}$ and $\beta_{\widetilde{\mathbf{t}}, x, \mathbf{t}}$ is the isomorphism in $\mathfrak{C}$ which is "responsible for putting tensor factors in the correct order".*

*Morphisms of pseudo-cooperads are defined in the obvious way.*

Similarly, reversing arrows in (2.32), (2.33), and Definition 2.2 we get the notion of counit and the definition of a $\Xi$-colored cooperad.

The $\Xi$-colored collection (2.35) carries a unique structure of a $\Xi$-colored cooperad. Furthermore, $*$ is the terminal object in the category of $\Xi$-colored cooperads.

[4] Although, in paper [27] the author considers only non-colored operad, the line of arguments can be easily extended to the colored setting.

Dually to augmentation we define a coaugmentation on a (colored) cooperad $Q$ as a cooperad morphism

$$\varepsilon' : * \to Q\,.$$

For every coaugmented (colored) cooperad $Q$ the cokernel of coaugmentation naturally forms a (colored) pseudo-cooperad. We denote this pseudo-cooperad by $Q_\circ$.

As well as for (colored) operads, the assignment

$$Q \rightsquigarrow Q_\circ$$

extends to a functor which establishes an equivalence between the category of coaugmented (colored) cooperads and the category of (colored) pseudo-cooperads.

### 2.3 The convolution Lie algebra

Let $\mathcal{C}$ (resp. $\mathcal{O}$) be a $\Xi$-colored pseudo-cooperad (resp. $\Xi$-colored pseudo-operad) in $\mathsf{Ch}_{\mathbb{K}}$

We consider the following cochain complex

$$\mathrm{Conv}(\mathcal{C}, \mathcal{O}) := \prod_{\mathbf{q}} \mathrm{Hom}_{S_{\mathbf{q}}}(\mathcal{C}(\mathbf{q}), \mathcal{O}(\mathbf{q}))\,, \tag{2.41}$$

where the product is taken over all $\Xi$-colored planar corollas.

Let us denote by $\mathsf{Isom}^{\Xi}_2(\mathbf{q})$ the set of isomorphism classes in[5] $\mathsf{Tree}^{\Xi}_2(\mathbf{q})$. Let us choose for every class $z \in \mathsf{Isom}^{\Xi}_2(\mathbf{q})$ its representative $\mathbf{t}_z$.

Using the trees $\mathbf{t}_z$ we equip the complex (2.41) with the following binary operation

$$f \bullet g(X) = - \sum_{z \in \mathsf{Isom}^{\Xi}_2(\mathbf{q})} \mu_{\mathbf{t}_z} \big( f \otimes g(\Delta_{\mathbf{t}_z}(X)) \big) \tag{2.42}$$

where $X \in \mathcal{C}(\mathbf{q})$. The axioms of pseudo-(co)operad imply that $\bullet$ is a well-defined operation. Namely, the right hand side of (2.42) does not depend on the choice of representatives $\mathbf{t}_z$ and $f \bullet g$ is $S_{\mathbf{q}}$-equivariant.

We claim that

**Proposition 2.1** *The operation* $\bullet$ (2.42) *equips* $\mathrm{Conv}(\mathcal{C}, \mathcal{O})$ *with a pre-Lie algebra structure. In other words,*

$$(f \bullet g) \bullet h - f \bullet (g \bullet h) = (-1)^{|g||h|} (f \bullet h) \bullet g - (-1)^{|g||h|} f \bullet (h \bullet g)\,, \tag{2.43}$$

*for all homogeneous vectors* $f, g, h \in \mathrm{Conv}(\mathcal{C}, \mathcal{O})$.

Proof. This statement was proved in the more general setting (for PROPs) in [29, Section 2.2] by B. Vallette and S. Merkulov. For non-colored (co)operads, a detailed proof can be found in [9]. $\square$

Proposition 2.1 implies that the operation

$$[f, g] = f \bullet g - (-1)^{|f||g|} g \bullet f \tag{2.44}$$

satisfies the Jacobi identity. Thus, $\mathrm{Conv}(\mathcal{C}, \mathcal{O})$ is a Lie algebra in the category $\mathsf{Ch}_{\mathbb{K}}$. Following [29], we call $\mathrm{Conv}(\mathcal{C}, \mathcal{O})$ the *convolution Lie algebra* of a pair $(\mathcal{C}, \mathcal{O})$.

[5] Recall that $\mathsf{Tree}^{\Xi}_2(\mathbf{q})$ is the full subcategory of $\mathsf{Tree}^{\Xi}(\mathbf{q})$ whose objects are labeled $\Xi$-colored planar trees $\mathbf{t}$ with exactly two internal vertices.

Using "arity" we can equip the convolution Lie algebra $\mathrm{Conv}(\mathcal{C},\mathcal{O})$ with the natural descending filtration

$$\mathrm{Conv}(\mathcal{C},\mathcal{O}) = \mathcal{F}_0\,\mathrm{Conv}(\mathcal{C},\mathcal{O}) \supset \mathcal{F}_1\,\mathrm{Conv}(\mathcal{C},\mathcal{O}) \supset \mathcal{F}_2\,\mathrm{Conv}(\mathcal{C},\mathcal{O}) \supset \dots\,,$$

where

$$\mathcal{F}_m\,\mathrm{Conv}(\mathcal{C},\mathcal{O}) = \tag{2.45}$$

$$\big\{ f \in \mathrm{Conv}(\mathcal{C},\mathcal{O}) \quad \big| \quad f\Big|_{\mathcal{C}(\mathbf{q})} = 0 \quad \forall \text{ corollas } \mathbf{q} \quad \text{satisfying } |\mathbf{q}| \le m \big\}\,,$$

where

$$|\mathbf{q}| = \sum_{\chi\in\Xi} |c^{-1}_{\mathbf{q},l}(\chi)|$$

i.e. $|\mathbf{q}|$ is the total number of external edges of the corolla $\mathbf{q}$.

It is easy to see that this filtration is compatible with the Lie bracket and $\mathrm{Conv}(\mathcal{C},\mathcal{O})$ is complete with respect to this filtration. Namely,

$$\mathrm{Conv}(\mathcal{C},\mathcal{O}) = \lim_{m} \mathrm{Conv}(\mathcal{C},\mathcal{O}) \Big/ \mathcal{F}_m\,\mathrm{Conv}(\mathcal{C},\mathcal{O})\,. \tag{2.46}$$

In fact, we may introduce an additional descending filtration $\mathcal{F}^{\chi}_{\bullet}$ on the convolution Lie algebra $\mathrm{Conv}(\mathcal{C},\mathcal{O})$ for each color $\chi \in \Xi$:

$$\mathrm{Conv}(\mathcal{C},\mathcal{O}) = \mathcal{F}^{\chi}_{-1}\,\mathrm{Conv}(\mathcal{C},\mathcal{O}) \supset \mathcal{F}^{\chi}_{0}\,\mathrm{Conv}(\mathcal{C},\mathcal{O}) \supset \mathcal{F}^{\chi}_{1}\,\mathrm{Conv}(\mathcal{C},\mathcal{O}) \supset \dots\,,$$

where

$$\mathcal{F}^{\chi}_m\,\mathrm{Conv}(\mathcal{C},\mathcal{O}) = \tag{2.47}$$

$$\big\{ f \in \mathrm{Conv}(\mathcal{C},\mathcal{O}) \quad \big| \quad f\Big|_{\mathcal{C}(\mathbf{q})} = 0 \quad \forall \text{ corollas } \mathbf{q} \quad \text{satisfying } |c^{-1}_{\mathbf{q},l}(\chi)| \le m \big\}\,.$$

In other words, a vector $f \in \mathrm{Conv}(\mathcal{C},\mathcal{O})$ belongs to $\mathcal{F}^{\chi}_m\mathrm{Conv}(\mathcal{C},\mathcal{O})$ if and only if

$$f(\mathbf{q}) = 0$$

for every $\Xi$-colored planar corolla $\mathbf{q}$ with at most $m$ external edges carrying the color $\chi$.

It is not hard to see that the filtration (2.47) is compatible with the Lie bracket on $\mathrm{Conv}(\mathcal{C},\mathcal{O})$ and $\mathrm{Conv}(\mathcal{C},\mathcal{O})$ is complete with respect to this filtration.

## 2.4 Free $\Xi$-colored operad

Let $Q$ be a $\Xi$-colored collection. Following [2], the spaces $\Psi\mathbb{OP}(Q)(\mathbf{q})$ of the free $\Xi$-colored pseudo-operad generated by the collection $Q$ are

$$\Psi\mathbb{OP}(Q)(\mathbf{q}) = \mathrm{colim}\, \underline{Q}\Big|_{\mathsf{Tree}^{\Xi}(\mathbf{q})}\,, \tag{2.48}$$

where $\mathsf{Tree}^{\Xi}(\mathbf{q})$ is the full subcategory of $\mathsf{Tree}^{\Xi}$ whose objects are labeled $\Xi$-colored planar trees $\mathbf{t}$ satisfying condition (2.16).

The pseudo-operad structure on $\Psi\mathbb{OP}(Q)$ is defined in the obvious way using grafting of trees.

The free $\Xi$-colored operad $\mathbb{OP}(Q)$ is obtained from $\Psi\mathbb{OP}(Q)$ via adjoining the units.

Unfolding (2.48) we see that $\Psi\mathbb{OP}(Q)(\mathbf{q})$ is the quotient of the direct sum

$$\bigoplus_{\mathbf{t},\kappa(\mathbf{t})=\mathbf{q}} \underline{Q}(\mathbf{t}) \tag{2.49}$$

by the subspace spanned by vectors of the form

$$(\mathbf{t}, X) - (\mathbf{t}', \underline{Q}(\lambda)(X))$$

where $\lambda : \mathbf{t} \to \mathbf{t}'$ is a morphism in $\mathsf{Tree}^{\Xi}(\mathbf{q})$ and $X \in \underline{Q}(\mathbf{t})$.

Thus it is convenient to represent vectors in $\Psi\mathbb{OP}(Q)$ and in $\mathbb{OP}(Q)$ by labeled $\Xi$-colored planar trees with internal vertices decorated by vectors in $Q$. The decoration is subject to this rule: if $\kappa(x)$ is the corolla formed by all edges adjacent to an internal vertex $x$ then $x$ is decorated by a vector $v_x \in Q(\kappa(x))$.

If a decorated tree $\mathbf{t}'$ is obtained from a decorated tree $\mathbf{t}$ by applying an element $\sigma \in S_{\kappa(x)}$ to incoming edges of a vertex $x$ and replacing the vector $v_x$ by $\sigma^{-1}(v_x)$ then $\mathbf{t}'$ and $\mathbf{t}$ represent the same vectors in (2.48).

**Example 2.7** Let $Q$ be a 2-colored collection. Figure 2.16 shows a labeled 2-colored tree $\mathbf{t}$ decorated by vectors $v_1 \in Q(1,2)^{\mathfrak{o}}$, $v_2 \in Q(2,0)^{\mathfrak{c}}$ and $v_3 \in Q(1,0)^{\mathfrak{o}}$. Figure 2.17 shows another decorated tree with $v_1' = (\mathrm{id}, \sigma_{12})(v_1)$ and $v_2' = \sigma_{12}(v_2)$, where $\sigma_{12}$ is the transposition in $S_2$. According to our discussion, these trees represent the same vector in $\mathbb{OP}(Q)(3,1)^{\mathfrak{o}}$.

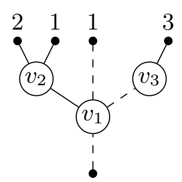



Fig. 2.16: A 2-colored decorated tree $\mathbf{t}$

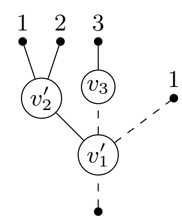



Fig. 2.17: A 2-colored decorated tree $\widetilde{\mathbf{t}}$. Here $v_1' = (\mathrm{id}, \sigma_{12})(v_1)$ and $v_2' = \sigma_{12}(v_2)$

## 2.5 The cobar construction in the colored setting

The cobar construction [11], [15], [16] is a functor from the category of coaugmented cooperad (in $\mathsf{Ch}_{\mathbb{K}}$) to the category of augmented operads (in $\mathsf{Ch}_{\mathbb{K}}$). It is used to construct free resolutions for operads. In this section we briefly describe the cobar construction in the colored setting.

Let $\mathcal{C}$ be a coaugmented $\Xi$-colored cooperad in the category $\mathsf{Ch}_{\mathbb{K}}$ and $\mathcal{C}_\circ$ be the cokernel of coaugmentation. As an operad in the category $\mathsf{grVect}_{\mathbb{K}}$, $\mathrm{Cobar}(\mathcal{C})$ is freely generated by the collection $\mathbf{s}\,\mathcal{C}_\circ$

$$\mathrm{Cobar}(\mathcal{C}) = \mathbb{OP}(\mathbf{s}\,\mathcal{C}_\circ). \tag{2.50}$$

Thus, it suffices to define the differential $\partial^{\mathrm{Cobar}}$ on generators $X \in \mathbf{s}\,\mathcal{C}_\circ$.

The differential $\partial^{\mathrm{Cobar}}$ on $\mathrm{Cobar}(\mathcal{C})$ can be written as the sum

$$\partial^{\mathrm{Cobar}} = \partial' + \partial'',$$

with

$$\partial'(X) = -\mathbf{s}\,\partial_{\mathcal{C}}\,\mathbf{s}^{-1}X \tag{2.51}$$

and

$$\partial''(X) = -\mathbf{s}\,\partial_{\mathcal{C}}\,\mathbf{s}^{-1}X + \bigoplus_{z \in \mathsf{Isom}^{\Xi}_{2}(\mathbf{q})} (\mathbf{s}\otimes\mathbf{s})\big(\mathbf{t}_z; \Delta_{\mathbf{t}_z}(\mathbf{s}^{-1}X)\big) \tag{2.52}$$

where $X \in \mathbf{s}\,\mathcal{C}_{\circ}(\mathbf{q})$, $\mathsf{Isom}^{\Xi}_{2}(\mathbf{q})$ is the set of isomorphism classes in $\mathsf{Tree}^{\Xi}_{2}(\mathbf{q})$, the tree $\mathbf{t}_z$ is any representative of the class $z$, and $\partial_{\mathcal{C}}$ is the differential on $\mathcal{C}$.

Properties of comultiplications $\Delta_{\mathbf{t}}$ imply that the right hand side of (2.52) does not depend on the choice of representatives $\mathbf{t}_z$. Furthermore, using the identity $\big(\partial_{\mathcal{C}}\big)^2 = 0$ and the compatibility of $\partial_{\mathcal{C}}$ with comultiplications $\Delta_{\mathbf{t}}$ one easily deduces that

$$\partial' \circ \partial' = 0\,,$$

and

$$\partial' \circ \partial'' + \partial'' \circ \partial' = 0\,.$$

Finally the coassociativity law (2.40) implies that

$$\partial'' \circ \partial'' = 0\,. \tag{2.53}$$

Let $\mathcal{O}$ be a $\Xi$-colored operad in $\mathsf{Ch}_{\mathbb{K}}$. We claim that

**Proposition 2.2** *For every coaugmented $\Xi$-colored cooperad $\Xi$ in $\mathsf{Ch}_{\mathbb{K}}$ operad morphisms from* $\mathrm{Cobar}(\mathcal{C})$ *to $\mathcal{O}$ are in bijection with MC elements of the Lie algebra*

$$\mathrm{Conv}(\mathcal{C}_{\circ}, \mathcal{O})\,, \tag{2.54}$$

*where $\mathcal{O}$ is viewed as a $\Xi$-colored pseudo-operad via the forgetful functor.*

Proof. Since $\mathrm{Cobar}(\mathcal{C})$ is freely generated by the $\Xi$-colored collection $\mathbf{s}\,\mathcal{C}_{\circ}$ any operad morphism

$$F : \mathrm{Cobar}(\mathcal{C}) \to \mathcal{O}$$

is uniquely determined by its restriction to $\mathbf{s}\,\mathcal{C}_{\circ}$.

Let us denote by $\alpha_F$ the degree 1 element

$$\alpha_F : \mathrm{Conv}(\mathcal{C}_{\circ}, \mathcal{O}) \tag{2.55}$$

corresponding to the restriction

$$F\Big|_{\mathbf{s}\,\mathcal{C}_{\circ}} : \mathbf{s}\,\mathcal{C}_{\circ} \to \mathcal{O}\,.$$

A direct computation shows that the compatibility of $F$ with the differentials is equivalent to the MC equation on $\alpha_F$ in the Lie algebra (2.54). $\square$

**Remark 2.8** It is possible to express the Lie bracket on $\mathrm{Conv}(\mathcal{C}_{\circ}, \mathcal{O})$ in terms of the portion $\partial''$ (2.52) of the cobar differential $\partial^{\mathrm{Cobar}}$. More precisely, for $f, g \in \mathrm{Conv}(\mathcal{C}_{\circ}, \mathcal{O})$ and $X \in \mathcal{C}_{\circ}$ we have

$$[f,g](X) = (-1)^{|g|}\mu\big(f\mathbf{s}^{-1} \otimes g\mathbf{s}^{-1}(\partial''(\mathbf{s}X))\big) - (-1)^{|f||g|}(f \leftrightarrow g)\,, \tag{2.56}$$

where $f\mathbf{s}^{-1}$ and $g\mathbf{s}^{-1}$ act in the obvious way on the tensor factors of $\partial''(\mathbf{s}X) \in \mathbb{OP}(\mathbf{s}\,\mathcal{C}_{\circ})$ and $\mu$ denotes the multiplication map

$$\mu : \mathbb{OP}(\mathcal{O}) \to \mathcal{O}\,.$$

# 3 Operad $\mathsf{dGra}$ and its 2-colored extension $\mathsf{KGra}$

Let us remind from [37] the operad (in $\mathsf{grVect}_{\mathbb{K}}$) of directed labeled graphs $\mathsf{dGra}$.

To define the space $\mathsf{dGra}(n)$ we introduce an auxiliary set $\mathrm{dgra}_n$. An element of $\mathrm{dgra}_n$ is a directed labelled graph $\Gamma$ with $n$ vertices and with the additional piece of data: the set of edges of $\Gamma$ is equipped with a total order. An example of an element in $\mathrm{dgra}_4$ is shown on figure 3.1.

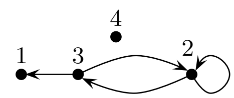



Fig. 3.1: The edges are equipped with the order $(3,1) < (3,2) < (2,3) < (2,2)$

The space $\mathsf{dGra}(n)$ is spanned by elements of $\mathrm{dgra}_n$, modulo the relation $\Gamma^{\sigma} = (-1)^{|\sigma|}\Gamma$ where the graphs $\Gamma^{\sigma}$ and $\Gamma$ correspond to the same directed labelled graph but differ only by permutation $\sigma$ of edges. We also declare that the degree of a graph $\Gamma$ in $\mathsf{dGra}(n)$ equals $-e(\Gamma)$, where $e(\Gamma)$ is the number of edges in $\Gamma$. For example, the graph $\Gamma$ on figure 3.1 has 4 edges. Thus its degree is $-4$.

**Remark 3.1** It clear that if a graph $\Gamma \in \mathrm{dgra}_n$ has multiple edges with the same direction (as in the graph on figure 3.2) then

$$\Gamma = -\Gamma$$

in $\mathsf{dGra}(n)$. Thus we may discard such graphs from the very beginning.

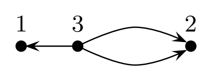



Fig. 3.2: Vertices 2 and 3 are connected by two edges with the same direction

## 3.1 Operad structure on $\mathsf{dGra}$

Let $\Gamma$ and $\widetilde{\Gamma}$ be graphs representing vectors in $\mathsf{dGra}(n)$ and $\mathsf{dGra}(m)$, respectively. Let $1 \le i \le m$.

The vector $\widetilde{\Gamma} \circ_i \Gamma \in \mathsf{dGra}(n+m-1)$ is represented by the sum of graphs $\Gamma_{\alpha} \in \mathrm{dgra}_{n+m-1}$

$$\widetilde{\Gamma} \circ_i \Gamma = \sum_{\alpha} \Gamma_{\alpha}\,, \tag{3.1}$$

where $\Gamma_{\alpha}$ is obtained by "plugging in" the graph $\Gamma$ into the $i$-th vertex of the graph $\widetilde{\Gamma}$ and reconnecting the edges incident to the $i$-th vertex of $\widetilde{\Gamma}$ to vertices of $\Gamma$ in all possible ways. (The index $\alpha$ refers to a particular way of connecting the edges incident to the $i$-th vertex of $\widetilde{\Gamma}$ to vertices of $\Gamma$. ) After reconnecting edges we label vertices of $\Gamma_{\alpha}$ as follows:

- first, we shift all labels on vertices of $\Gamma$ up by $i-1$;
- second, we shift the labels on the last $m-i$ vertices of $\widetilde{\Gamma}$ up by $n-1$.

To define the total order on edges of the graph $\Gamma_\alpha$ we declare that all edges of $\widetilde{\Gamma}$ are smaller than all edges of the graph $\Gamma$.

**Example 3.2** Let $\widetilde{\Gamma}$ (resp. $\Gamma$) be the graph depicted on figure 3.3 (resp. figure 3.4). The vector $\widetilde{\Gamma} \circ_2 \Gamma$ is shown on figure 3.5. For the first graph in the sum $\widetilde{\Gamma} \circ_2 \Gamma$ we have $(1,2) <$

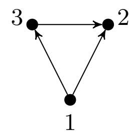



Fig. 3.3: A graph $\widetilde{\Gamma} \in \mathrm{dgra}_3$. The order on edges is $(1,2) < (1,3) < (3,2)$

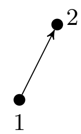



Fig. 3.4: A graph $\Gamma \in \mathrm{dgra}_2$

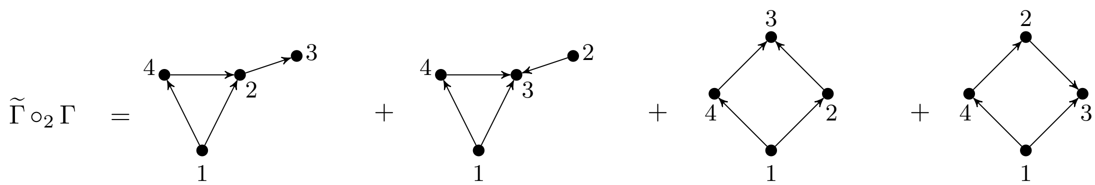



Fig. 3.5: The vector $\widetilde{\Gamma} \circ_2 \Gamma \in \mathsf{dGra}(4)$

$(1,4) < (4,2) < (2,3)$. For the second graph in the sum $\widetilde{\Gamma} \circ_2 \Gamma$ we have $(1,3) < (1,4) < (4,3) < (2,3)$. For the third graph in the sum $\widetilde{\Gamma} \circ_2 \Gamma$ we have $(1,2) < (1,4) < (4,3) < (2,3)$. Finally, for the last graph in the sum $\widetilde{\Gamma} \circ_2 \Gamma$ we have $(1,3) < (1,4) < (4,2) < (2,3)$.

The symmetric group $S_n$ acts on $\mathsf{dGra}(n)$ in the obvious way by rearranging the labels on vertices. It is not hard to see that insertions (3.1) together with this action of $S_n$ give on $\mathsf{dGra}$ an operad structure with the identity element being the unique graph in $\mathrm{dgra}_1$ with no edges.

### 3.2 2-colored operad $\mathsf{KGra}$

To define a stable formality quasi-isomorphism we need to upgrade the operad $\mathsf{dGra}$ to a 2-colored operad $\mathsf{KGra}$ (in $\mathsf{grVect}_{\mathbb{K}}$). The additional spaces of the operad $\mathsf{KGra}$ are assembled from the graphs which were used by M. Kontsevich in his groundbreaking paper [24]. As far as I understand, T. Willwacher is using this operad in [38] under the different name: $\mathsf{SGra}$.

Recall that, following our conventions, $\mathsf{KGra}(n,k)^{\mathfrak{c}}$ denotes the space of operations with $n$ inputs of color $\mathfrak{c}$, $k$ inputs of color $\mathfrak{o}$, and with the color of the output being $\mathfrak{c}$. Similarly, $\mathsf{KGra}(n,k)^{\mathfrak{o}}$ is the space of operations with $n$ inputs of color $\mathfrak{c}$, $k$ inputs of color $\mathfrak{o}$, and with the color of the output being $\mathfrak{o}$.

First, we declare that $\mathsf{KGra}(n,k)^{\mathfrak{c}} = \mathbf{0}$ whenever $k \ge 1$.

Next, for the space $\mathsf{KGra}(n,0)^{\mathfrak{c}}$ ($n \ge 0$) we have

$$\mathsf{KGra}(n,0)^{\mathfrak{c}} = \mathsf{dGra}(n). \tag{3.2}$$

To define the space $\mathsf{KGra}(n,k)^{\mathfrak{o}}$ we introduce the auxiliary set $\mathrm{dgra}_{n,k}$. An element of the set $\mathrm{dgra}_{n,k}$ is a directed labelled graph $\Gamma$ with $n$ vertices of color $\mathfrak{c}$, $k$ vertices of color $\mathfrak{o}$, and

with the following data: the set of edges of $\Gamma$ is equipped with a total order. In addition, we require that each graph $\Gamma \in \mathrm{dgra}_{n,k}$ has no edges originating from any vertex with color $\mathfrak{o}$.

**Example 3.3** Figure 3.6 shows an example of a graph in $\mathrm{dgra}_{2,3}$. Black (resp. white) vertices carry the color $\mathfrak{c}$ (resp. $\mathfrak{o}$). We use separate labels for vertices of color $\mathfrak{c}$ and vertices of color $\mathfrak{o}$. For example $2_{\mathfrak{c}}$ denotes the vertex of color $\mathfrak{c}$ with label 2 and $3_{\mathfrak{o}}$ denotes the vertex of color $\mathfrak{o}$ with label 3.

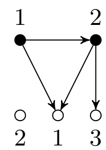



Fig. 3.6: We equip the edges with the order $(1_{\mathfrak{c}}, 2_{\mathfrak{c}}) < (1_{\mathfrak{c}}, 1_{\mathfrak{o}}) < (2_{\mathfrak{c}}, 1_{\mathfrak{o}}) < (2_{\mathfrak{c}}, 3_{\mathfrak{o}})$

The space $\mathsf{KGra}(n,k)^{\mathfrak{o}}$ is spanned by elements of $\mathrm{dgra}_{n,k}$, modulo the relation $\Gamma^{\sigma} = (-1)^{|\sigma|}\Gamma$ where the graphs $\Gamma^{\sigma}$ and $\Gamma$ correspond to the same directed labelled graph but differ only by permutation $\sigma$ of edges. As above, we declare that the degree of a graph $\Gamma$ in $\mathsf{KGra}(n,k)^{\mathfrak{o}}$ equals $-e(\Gamma)$.

The elementary insertions

$$\mathsf{KGra}(m,0)^{\mathfrak{c}} \otimes \mathsf{KGra}(n,0)^{\mathfrak{c}} \to \mathsf{KGra}(m+n-1,0)^{\mathfrak{c}}$$

are defined in the same way as for $\mathsf{dGra}$. So we proceed to the remaining insertions.

### 3.3 Elementary insertions $\mathsf{KGra}(m,k)^{\mathfrak{o}} \otimes \mathsf{KGra}(n,0)^{\mathfrak{c}} \to \mathsf{KGra}(m+n-1,k)^{\mathfrak{o}}$

Let $\Gamma$ and $\widetilde{\Gamma}$ be graphs representing vectors in $\mathsf{KGra}(n,0)^{\mathfrak{c}}$ and $\mathsf{KGra}(m,k)^{\mathfrak{o}}$, respectively. Let $1 \le i \le m$.

The vector $\widetilde{\Gamma} \circ_{i,\mathfrak{c}} \Gamma \in \mathsf{KGra}(n+m-1,k)^{\mathfrak{o}}$ is the sum of graphs $\Gamma_{\alpha} \in \mathrm{dgra}_{n+m-1,k}$

$$\widetilde{\Gamma} \circ_{i,\mathfrak{c}} \Gamma = \sum_{\alpha} \Gamma_{\alpha}\,, \tag{3.3}$$

where $\Gamma_{\alpha}$ is obtained by "plugging in" the graph $\Gamma$ into the $i$-th black vertex of the graph $\widetilde{\Gamma}$ and reconnecting the edges incident to this vertex to vertices of $\Gamma$ in all possible ways. (The index $\alpha$ refers to a particular way of connecting the edges incident to the $i$-th black vertex of $\widetilde{\Gamma}$ to vertices of $\Gamma$. ) After reconnecting edges we label vertices of $\Gamma_{\alpha}$ as follows:

- first, we shift all labels on vertices of $\Gamma$ up by $i-1$;
- second, we shift labels on the last $m-i$ black vertices of $\widetilde{\Gamma}$ up by $n-1$.

To define the total order on edges of the graph $\Gamma_{\alpha}$ we declare that all edges of $\widetilde{\Gamma}$ are smaller than all edges of the graph $\Gamma$.

**Example 3.4** The graphs depicted on figures 3.7 and 3.8 represent vectors $\widetilde{\Gamma} \in \mathsf{KGra}(2,1)^{\mathfrak{o}}$ and $\Gamma \in \mathsf{KGra}(2,0)^{\mathfrak{c}}$, respectively. For the edges of $\widetilde{\Gamma}$ we set

$$(1_{\mathfrak{c}}, 1_{\mathfrak{o}}) < (1_{\mathfrak{c}}, 2_{\mathfrak{c}}) < (2_{\mathfrak{c}}, 1_{\mathfrak{o}})\,.$$

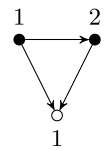


Fig. 3.7: The graph $\widetilde{\Gamma}$

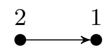


Fig. 3.8: The graph $\Gamma$

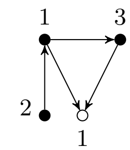


Fig. 3.9: The graph $\Gamma_1$

As above, white vertices carry the color $\mathfrak{o}$ and black vertices carry the color $\mathfrak{c}$.

The vector $\widetilde{\Gamma} \circ_1 \Gamma \in \mathsf{KGra}(3,1)^{\mathfrak{o}}$ is represented by the sum of graphs $\Gamma_1$, $\Gamma_2$, $\Gamma_3$, $\Gamma_4$ depicted on figures 3.9, 3.10, 3.11, 3.12, respectively. Following our rule, the edges of $\Gamma_1$ are ordered

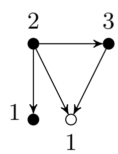


Fig. 3.10: The graph $\Gamma_2$

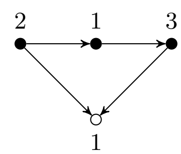


Fig. 3.11: The graph $\Gamma_3$

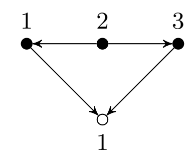


Fig. 3.12: The graph $\Gamma_4$

as follows $(1_{\mathfrak{c}}, 1_{\mathfrak{o}}) < (1_{\mathfrak{c}}, 3_{\mathfrak{c}}) < (3_{\mathfrak{c}}, 1_{\mathfrak{o}}) < (2_{\mathfrak{c}}, 1_{\mathfrak{c}})$. Similarly, edges of $\Gamma_2$ carry the order $(2_{\mathfrak{c}}, 1_{\mathfrak{o}}) < (2_{\mathfrak{c}}, 3_{\mathfrak{c}}) < (3_{\mathfrak{c}}, 1_{\mathfrak{o}}) < (2_{\mathfrak{c}}, 1_{\mathfrak{c}})$. The edges of $\Gamma_3$ are equipped with the order $(2_{\mathfrak{c}}, 1_{\mathfrak{o}}) < (1_{\mathfrak{c}}, 3_{\mathfrak{c}}) < (3_{\mathfrak{c}}, 1_{\mathfrak{o}}) < (2_{\mathfrak{c}}, 1_{\mathfrak{c}})$. Finally, for $\Gamma_4$ we have $(1_{\mathfrak{c}}, 1_{\mathfrak{o}}) < (2_{\mathfrak{c}}, 3_{\mathfrak{c}}) < (3_{\mathfrak{c}}, 1_{\mathfrak{o}}) < (2_{\mathfrak{c}}, 1_{\mathfrak{c}})$.

### 3.4 Elementary insertions $\mathsf{KGra}(m,p)^{\mathfrak{o}} \otimes \mathsf{KGra}(n,q)^{\mathfrak{o}} \to \mathsf{KGra}(m+n, p+q-1)^{\mathfrak{o}}$

Let $\Gamma$ and $\widetilde{\Gamma}$ be graphs representing vectors in $\mathsf{KGra}(n,q)^{\mathfrak{o}}$ and $\mathsf{KGra}(m,p)^{\mathfrak{o}}$, respectively. Let $1 \le i \le p$.

The vector $\widetilde{\Gamma} \circ_{i,\mathfrak{o}} \Gamma \in \mathsf{KGra}(m+n, p+q-1)^{\mathfrak{o}}$ is represented by the sum of graphs $\Gamma_\alpha \in \mathrm{dgra}_{m+n,p+q-1}$

$$\widetilde{\Gamma} \circ_{i,\mathfrak{o}} \Gamma = \sum_\alpha \Gamma_\alpha \,, \tag{3.4}$$

where $\Gamma_\alpha$ is obtained by "plugging in" the graph $\Gamma$ into the $i$-th white vertex of the graph $\widetilde{\Gamma}$ and reconnecting the edges incident to this vertex to vertices of $\Gamma$ in all possible ways. (The index $\alpha$ refers to a particular way of connecting the edges incident to the $i$-th white vertex of $\widetilde{\Gamma}$ to vertices of $\Gamma$. ) After reconnecting edges we label vertices of $\Gamma_\alpha$ as follows:

- we shift all labels on black vertices of $\Gamma$ up by $m$;
- we shift all labels on white vertices of $\Gamma$ up by $i-1$;
- finally, we shift labels on the last $p-i$ white vertices of $\widetilde{\Gamma}$ up by $q-1$.

To define the total order on edges of the graph $\Gamma_\alpha$ we declare that all edges of $\widetilde{\Gamma}$ are smaller than all edges of the graph $\Gamma$.

**Example 3.5** If $\widetilde{\Gamma}$ is the graph depicted on figure 3.6 and $\Gamma$ is the graph depicted on figure 3.13 then the vector $\widetilde{\Gamma} \circ_{3,\mathfrak{o}} \Gamma \in \mathsf{KGra}(3,3)$ is the sum of graphs depicted on figure 3.14. For the edges of the first graph in this sum we have $(1_{\mathfrak{c}}, 2_{\mathfrak{c}}) < (1_{\mathfrak{c}}, 1_{\mathfrak{o}}) < (2_{\mathfrak{c}}, 1_{\mathfrak{o}}) < (2_{\mathfrak{c}}, 3_{\mathfrak{o}}) < (3_{\mathfrak{c}}, 3_{\mathfrak{o}})$. For the edges of the second graph in this sum we have $(1_{\mathfrak{c}}, 2_{\mathfrak{c}}) < (1_{\mathfrak{c}}, 1_{\mathfrak{o}}) < (2_{\mathfrak{c}}, 1_{\mathfrak{o}}) < (2_{\mathfrak{c}}, 3_{\mathfrak{c}}) < (3_{\mathfrak{c}}, 3_{\mathfrak{o}})$.

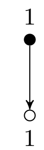


Fig. 3.13: A graph $\Gamma \in \mathrm{dgra}_{1,1}$

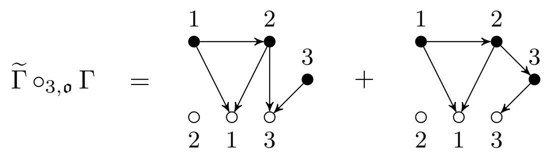


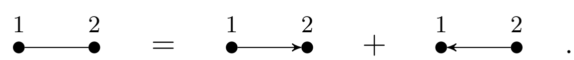
Fig. 3.14: The vector $\widetilde{\Gamma} \circ_{3,\mathfrak{o}} \Gamma$

The identity element $\mathbf{u}_{\mathfrak{c}} \in \mathsf{KGra}(1,0)^{\mathfrak{c}}$ (resp. $\mathbf{u}_{\mathfrak{o}} \in \mathsf{KGra}(0,1)^{\mathfrak{o}}$) is represented by the graph in $\mathrm{dgra}_1$ (resp. the graph in $\mathrm{dgra}_{0,1}$) with no edges.

It is straightforward to verify that $\mathbf{u}_{\mathfrak{c}}$, $\mathbf{u}_{\mathfrak{o}}$, and equations (3.1), (3.3), (3.4) together with the natural action of $S_n \times S_k$ on $\mathsf{KGra}(n,k)^{\mathfrak{o}}$ (resp. $S_n$ on $\mathsf{KGra}(n,0)^{\mathfrak{c}}$) define a structure of a 2-colored operad on $\mathsf{KGra}$ in $\mathsf{grVect}_{\mathbb{K}}$ .

**Remark 3.6** Let $\Gamma$ be a graph in $\mathrm{dgra}_n$ (resp. $\mathrm{dgra}_{n,k}$) and $e$ be edges of $\Gamma$ which connects two black vertices. We denote by $f_e(\Gamma)$ the graph which is obtained from $\Gamma$ by changing the direction of the edge $e$.

It is convenient to draw the linear combination $\Gamma + f_e(\Gamma)$ as a graph which is obtained from $\Gamma$ by forgetting the direction of $e$ . For example,

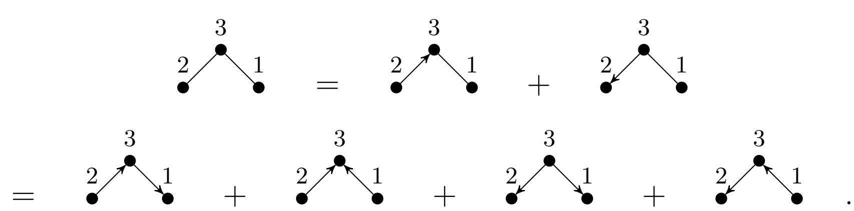


(3.5)

Similarly, if $e_1, e_2, \dots, e_p$ are edges of $\Gamma$ which connect only black vertices and the graph $\Gamma'$ is obtained from $\Gamma$ by forgetting the directions of the edges $e_1, e_2, \dots, e_p$, then $\Gamma'$ denotes the sum

$$\Gamma' = \sum_{k_i \in \{0,1\}} (f_{e_1})^{k_1} (f_{e_2})^{k_2} \dots (f_{e_p})^{k_p} (\Gamma)\,.$$

For example,



(3.6)

## 3.5 The action of the operad $\mathsf{KGra}$ on polyvectors and functions

Let $A$ be a free finitely generated commutative algebra (with unit) in $\mathsf{grVect}_{\mathbb{K}}$. We denote by

$$x^1, x^2, \dots, x^d \tag{3.7}$$

generators of $A$ and by $|x^1|, |x^2|, \dots, |x^d|$ their corresponding degrees. We think of $A$ as the algebra of functions on a graded affine space.

Let us denote by $V_A$ the free commutative algebra in $\mathsf{grVect}_{\mathbb{K}}$ generated by

$$x^1, x^2, \dots, x^d, \theta_1, \theta_2, \dots, \theta_d\,, \tag{3.8}$$

where $\theta_c$ carries the degree $|x^c| + 1$. We think of $V_A$ as the algebra of polyvector fields the corresponding graded affine space.

If all generators $x^c$ have degree 0 then $A$ (resp. $V_A$) is the algebra of functions (resp. the algebra of polyvector fields) on the affine space $\mathbb{K}^d$. However, for our constructions there is no need to impose any restrictions on degrees of generators (3.7).

We claim that

**Proposition 3.1** *The pair $(V_A, A)$ is naturally an algebra over the 2-colored operad* $\mathsf{KGra}$.

Proof. For $\Gamma \in \mathrm{dgra}_n$ and $v_1, v_2, \dots, v_n \in V_A$ we set

$$\Gamma(v_1, v_2, \dots, v_n) = \mathrm{mult}_n\Big(\Big[\prod_{(i,j)\in E(\Gamma)} \underline{\Delta}_{(i,j)}\Big]\,(v_1 \otimes v_2 \otimes \cdots \otimes v_n)\Big)\,, \tag{3.9}$$

where $\mathrm{mult}_n$ is the multiplication map

$$\mathrm{mult}_n : \big(V_A\big)^{\otimes\, n} \to V_A,$$

$$\underline{\Delta}_{(i,j)} = \sum_{c=1}^{d} 1 \otimes \cdots \otimes 1 \otimes \underbrace{\partial_{\theta_c}}_{i\text{-th slot}} \otimes 1 \otimes \cdots \otimes 1 \otimes \underbrace{\partial_{x^c}}_{j\text{-th slot}} \otimes 1 \otimes \cdots \otimes 1 \tag{3.10}$$

if $i < j$,

$$\underline{\Delta}_{(i,j)} = \sum_{c=1}^{d} 1 \otimes \cdots \otimes 1 \otimes \underbrace{\partial_{x^c}}_{j\text{-th slot}} \otimes 1 \otimes \cdots \otimes 1 \otimes \underbrace{\partial_{\theta_c}}_{i\text{-th slot}} \otimes 1 \otimes \cdots \otimes 1 \tag{3.11}$$

if $j < i$,

$$\underline{\Delta}_{(i,j)} = \sum_{c=1}^{d} 1 \otimes \cdots \otimes 1 \otimes \underbrace{\partial_{\theta_c}\partial_{x^c}}_{i\text{-th slot}} \otimes 1 \otimes \cdots \otimes 1 \tag{3.12}$$

if $i = j$, and the order of factors in the product

$$\prod_{(i,j)\in E(\Gamma)} \underline{\Delta}_{(i,j)}$$

comes from the order on the set $E(\Gamma)$ of edges of $\Gamma$.

To define the action of a graph $\Gamma \in \mathrm{dgra}_{n,k}$ we identify vertices of $\Gamma$ with the numbers $1, 2, \dots, n+k$ by using the labels and declaring that all black vertices precede all white vertices. Namely, the black vertex with label $i$ is identified with number $i$ and the white vertex with label $j$ is identified with number $n + j$. Then for $v_1, v_2, \dots, v_n \in V_A$, and $a_1, a_2, \dots, a_k \in A$ we set

$$\begin{aligned}&\Gamma(v_1, v_2, \dots, v_n; a_1, a_2, \dots, a_k)\\ &= \mathrm{mult}_{n,k}\Big(\Big[\prod_{(i,j)\in E(\Gamma)} \underline{\Delta}_{(i,j)}\Big]\,(v_1 \otimes v_2 \otimes \cdots \otimes v_n \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_k)\Big)\Big|_{\theta_c = 0}\,,\end{aligned} \tag{3.13}$$

where $\mathrm{mult}_{n,k}$ is the multiplication map

$$\mathrm{mult}_{n,k} : \big(V_A\big)^{\otimes n} \; \otimes \; A^{\otimes k} \to V_A \,,$$

$\underline{\Delta}_{(i,j)}$ is define by equations (3.10), (3.11), (3.12), and the order of factors in the product

$$\prod_{(i,j)\in E(\Gamma)} \underline{\Delta}_{(i,j)}$$

comes from the order on the set $E(\Gamma)$ of edges of $\Gamma$.

It is not hard to verify that equations (3.9), (3.13) define an action of $\mathsf{KGra}$ on the pair $(V_A, A)$.

$\square$

# 4 2-colored operad $\mathsf{OC}$ of H. Kajiura and J. Stasheff

Inspired by Zwiebach's open-closed string field theory [39], H. Kajiura and J. Stasheff introduced in [21] open-closed homotopy algebras (OCHA).

An OCHA is a pair of cochain complexes $(\mathcal{V}, \mathcal{A})$ with the following data:

- A $\Lambda\mathsf{Lie}_\infty$-structure on $\mathcal{V}$,
- an $A_\infty$-structure on $\mathcal{A}$, and
- a $\Lambda\mathsf{Lie}_\infty$-morphism from $\mathcal{V}$ to the Hochschild cochain complex $C^\bullet(\mathcal{A})$ of $\mathcal{A}$.

It was shown in [22], that OCHAs are governed by a 2-colored operad (in $\mathsf{Ch}_{\mathbb{K}}$) which we denote by $\mathsf{OC}$. Moreover, as an operad in $\mathsf{grVect}$, $\mathsf{OC}$ is freely generated by the 2-colored collection $\mathfrak{oc}$ with the following spaces:

$$\mathfrak{oc}(n,0)^{\mathfrak{c}} = \mathbf{s}^{3-2n}\mathbb{K}\,, \qquad n \ge 2\,, \tag{4.1}$$

$$\mathfrak{oc}(0,k)^{\mathfrak{o}} = \mathbf{s}^{2-k}\,\mathrm{sgn}_k \otimes \mathbb{K}[S_k]\,, \qquad k \ge 2\,, \tag{4.2}$$

$$\mathfrak{oc}(n,k)^{\mathfrak{o}} = \mathbf{s}^{2-2n-k}\,\mathrm{sgn}_k \otimes \mathbb{K}[S_k]\,, \qquad n \ge 1\,, \qquad k \ge 0\,, \tag{4.3}$$

where $\mathrm{sgn}_k$ is the sign representation of $S_k$. The remaining spaces of the collection $\mathfrak{oc}$ are zero.

Following the description of free colored operads via decorated (and colored) trees (see Section 2.4), we represent generators of $\mathsf{OC}$ in $\mathfrak{oc}(n,0)^{\mathfrak{c}}$ by non-planar labeled corollas with $n$ solid incoming edges (see figure 4.1). We represent generators of $\mathsf{OC}$ in $\mathfrak{oc}(0,k)^{\mathfrak{o}}$ by planar labeled corollas with $k$ dashed incoming edges (see figure 4.2). Finally, we use labeled 2-colored corollas with a planar structure given only on the dashed edges to represent generators of $\mathsf{OC}$ in $\mathfrak{oc}(n,k)$ (see figure 4.3).

Applying element $\sigma \in S_k$ to the labeled corolla $\mathbf{t}^{\mathfrak{o}}_k$ depicted on figure 4.2 we get a basis for the vector space $\mathfrak{oc}(0,k)^{\mathfrak{o}}$. Similarly, applying elements of $(\mathrm{id},\sigma) \in S_n \times S_k$ to the labeled corolla depicted on figure 4.3 we get a basis for the vector space $\mathfrak{oc}(n,k)^{\mathfrak{o}}$.

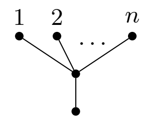



Fig. 4.1: The non-planar corolla $\mathbf{t}_n^{\mathfrak{c}}$ representing a generator of $\mathfrak{oc}(n,0)^{\mathfrak{c}}$

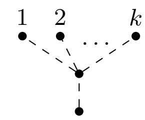



Fig. 4.2: The 2-colored planar corola $\mathbf{t}_k^{\mathfrak{o}}$ representing a generator of $\mathfrak{oc}(0,k)^{\mathfrak{o}}$

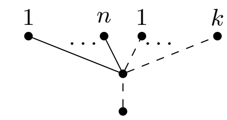



Fig. 4.3: The 2-colored partially planar corola $\mathbf{t}_{n,k}^{\mathfrak{o}}$ representing a generator of $\mathfrak{oc}(n,k)^{\mathfrak{o}}$

Equations (4.1), (4.2), and (4.3) imply that the corollas $\mathbf{t}_n^{\mathfrak{c}}$, $\mathbf{t}_k^{\mathfrak{o}}$ and $\mathbf{t}_{n,k}^{\mathfrak{o}}$ carry the following degrees:

$$|\mathbf{t}_n^{\mathfrak{c}}| = 3 - 2n \qquad n \ge 2\,, \tag{4.4}$$

$$|\mathbf{t}_k^{\mathfrak{o}}| = 2 - k \qquad k \ge 2\,, \tag{4.5}$$

$$|\mathbf{t}_{n,k}^{\mathfrak{o}}| = 2 - 2n - k \qquad n \ge 1,\ k \ge 0\,. \tag{4.6}$$

## 4.1 The differential on $\mathsf{OC}$

It is convenient to split the differential $\mathcal{D}$ on $\mathsf{OC}$ into four summands

$$\mathcal{D} = \mathcal{D}_{\mathsf{Lie}} + \mathcal{D}_{\mathsf{As}} + \mathcal{D}' + \mathcal{D}''\,. \tag{4.7}$$

Since $\mathsf{OC}$ is freely generated by the 2-colored collection $\mathfrak{oc}$, it suffices to define the values of summands $\mathcal{D}_{\mathsf{Lie}}$, $\mathcal{D}_{\mathsf{As}}$, $\mathcal{D}'$, and $\mathcal{D}''$ on corollas $\mathbf{t}_n^{\mathfrak{c}}$, $\mathbf{t}_k^{\mathfrak{o}}$, and $\mathbf{t}_{n,k}^{\mathfrak{o}}$ depicted on figures 4.1, 4.2, and 4.3, respectively.

For the corolla $\mathbf{t}_n^{\mathfrak{c}}$ we have

$$\mathcal{D}_{\mathsf{As}}(\mathbf{t}_n^{\mathfrak{c}}) = 0\,, \qquad \mathcal{D}'(\mathbf{t}_n^{\mathfrak{c}}) = 0\,, \qquad \mathcal{D}''(\mathbf{t}_n^{\mathfrak{c}}) = 0 \tag{4.8}$$

and $\mathcal{D}_{\mathsf{Lie}}(\mathbf{t}_n^{\mathfrak{c}})$ is the sum shown on figure 4.4.

$$\mathcal{D}_{\mathsf{Lie}}(\mathbf{t}_n^{\mathfrak{c}}) \quad = \quad - \sum_{p=2}^{n-1} \sum_{\tau \in \mathrm{Sh}_{p,n-p}}$$

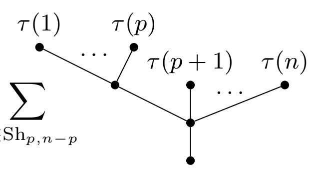



Fig. 4.4: The value of $\mathcal{D}_{\mathsf{Lie}}$ on $\mathbf{t}_n^{\mathfrak{c}}$

For the corolla $\mathbf{t}_k^{\mathfrak{o}}$ we have

$$\mathcal{D}_{\mathsf{Lie}}(\mathbf{t}_k^{\mathfrak{o}}) = 0\,, \qquad \mathcal{D}'(\mathbf{t}_k^{\mathfrak{o}}) = 0\,, \qquad \mathcal{D}''(\mathbf{t}_k^{\mathfrak{o}}) = 0\,, \tag{4.9}$$

and $\mathcal{D}_{\mathsf{As}}(\mathbf{t}_k^{\mathfrak{o}})$ is the sum shown on figure 4.5.

The value of $\mathcal{D}_{\mathsf{Lie}}$ on the corolla $\mathbf{t}_{n,k}^{\mathfrak{o}}$ is given by the sum depicted on figure 4.6 and the value of $\mathcal{D}_{\mathsf{As}}$ on the corolla $\mathbf{t}_{n,k}^{\mathfrak{o}}$ is given by the sum depicted on figure 4.7. The values $\mathcal{D}'(\mathbf{t}_{n,k}^{\mathfrak{o}})$ and $\mathcal{D}''(\mathbf{t}_{n,k}^{\mathfrak{o}})$ for $n \ge 2$ are defined on figures 4.8 and 4.9, respectively. Finally, for the corollas $\mathbf{t}_{1,k}^{\mathfrak{o}}$ we have

$$\mathcal{D}'(\mathbf{t}_{1,k}^{\mathfrak{o}}) = \mathcal{D}''(\mathbf{t}_{1,k}^{\mathfrak{o}}) = 0\,, \qquad \forall\ k \ge 0\,. \tag{4.10}$$

$$\mathcal{D}_{\mathsf{As}}(\mathbf{t}_k^{\mathsf{o}}) \quad = \quad -\sum_{p=0}^{k-2}\sum_{q=p+2}^{k} \quad (-1)^{p+(k-q)(q-p)}$$

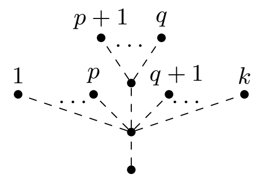



Fig. 4.5: The value of $\mathcal{D}_{\mathsf{As}}$ on $\mathbf{t}_k^{\mathsf{o}}$

$$\mathcal{D}_{\mathsf{Lie}}(\mathbf{t}_{n,k}^{\mathsf{o}}) \quad = \quad (-1)^k \quad \sum_{p=2}^{n-1}\sum_{\tau\in \mathrm{Sh}_{p,n-p}}$$

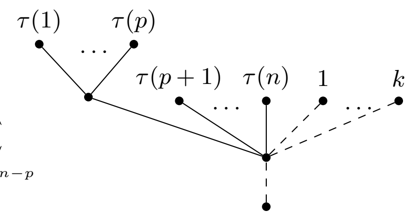
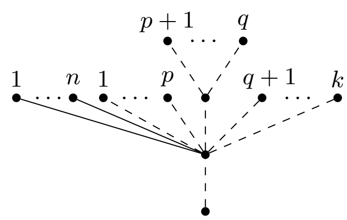



Fig. 4.6: The value of $\mathcal{D}_{\mathsf{Lie}}$ on $\mathbf{t}_{n,k}^{\mathsf{o}}$

$$\mathcal{D}_{\mathsf{As}}(\mathbf{t}_{n,k}^{\mathsf{o}}) \quad = \quad -\sum_{p=0}^{k-2}\sum_{q=p+2}^{k} \quad (-1)^{p+(k-q)(q-p)}$$




$$- \quad \sum_{p=0}^{k-2}\sum_{q=p+2}^{k} \quad (-1)^{p+(k-q)(q-p)}$$

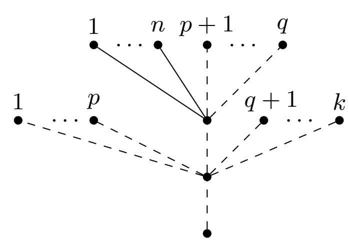



Fig. 4.7: The value of $\mathcal{D}_{\mathsf{As}}$ on $\mathbf{t}_{n,k}^{\mathsf{o}}$

$$\mathcal{D}'(\mathbf{t}_{n,k}^{\mathsf{o}}) \quad = \quad (-1)^k$$

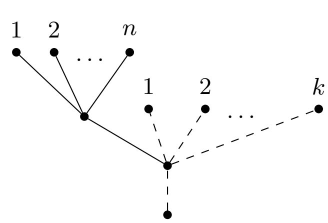



Fig. 4.8: The value of $\mathcal{D}'$ on $\mathbf{t}_{n,k}^{\mathsf{o}}$ for $n \geq 2$

$$\mathcal{D}''(\mathbf{t}^{\mathfrak{o}}_{n,k}) \quad = \quad - \sum_{r=1}^{n-1} \sum_{\sigma \in \mathrm{Sh}_{r,n-r}} \sum_{0 \le p \le q \le k} (-1)^{p+(k-q)(q-p)}$$

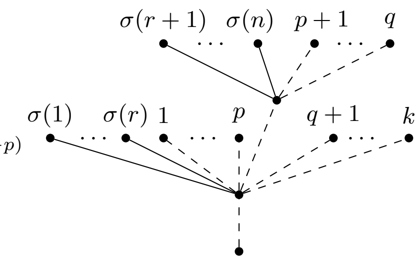



Fig. 4.9: The value of $\mathcal{D}''$ on $\mathbf{t}^{\mathfrak{o}}_{n,k}$ for $n \ge 2$

Direct computations show that

$$\big(\mathcal{D}_{\mathsf{As}}\big)^2 = 0\,, \tag{4.11}$$

$$\big(\mathcal{D}_{\mathsf{Lie}} + \mathcal{D}'\big)^2 = 0\,, \tag{4.12}$$

$$\mathcal{D}_{\mathsf{As}} \circ (\mathcal{D}_{\mathsf{Lie}} + \mathcal{D}') + (\mathcal{D}_{\mathsf{Lie}} + \mathcal{D}') \circ \mathcal{D}_{\mathsf{As}} = 0\,, \tag{4.13}$$

$$(\mathcal{D}_{\mathsf{Lie}} + \mathcal{D}') \circ \mathcal{D}'' + \mathcal{D}'' \circ (\mathcal{D}_{\mathsf{Lie}} + \mathcal{D}') = 0\,, \tag{4.14}$$

$$\mathcal{D}_{\mathsf{As}} \circ \mathcal{D}'' + \mathcal{D}'' \circ \mathcal{D}_{\mathsf{As}} + \mathcal{D}'' \circ \mathcal{D}'' = 0\,. \tag{4.15}$$

**Remark 4.1** It is not hard to see that the differential $\mathcal{D}$ on $\mathbb{OP}(\mathfrak{oc})$ defines on $\mathbf{s}^{-1}\mathfrak{oc}$ a structure of 2-colored pseudo-cooperad. Thus, if $\mathfrak{oc}^{\vee}$ is the 2-colored cooperad obtained from $\mathbf{s}^{-1}\mathfrak{oc}$ via formally adjoining the counit, then[6]

$$\mathsf{OC} = \mathrm{Cobar}(\mathfrak{oc}^{\vee})\,. \tag{4.16}$$

We remark that

$$\mathfrak{oc}^{\vee}(n,0)^{\mathfrak{c}} = \Lambda^2 \mathsf{coCom}(n) \tag{4.17}$$

and

$$\mathfrak{oc}^{\vee}(0,k)^{\mathfrak{o}} = \Lambda \mathsf{coAs}(k)\,. \tag{4.18}$$

## 4.2 $\mathsf{OC}$-algebras

As we stated above, an $\mathsf{OC}$-algebra is a pair of cochain complexes $(\mathcal{V}, \mathcal{A})$ with the following data:

- A $\Lambda\mathsf{Lie}_{\infty}$-structure on $\mathcal{V}$,
- an $A_{\infty}$-structure on $\mathcal{A}$, and
- a $\Lambda\mathsf{Lie}_{\infty}$-morphism from $\mathcal{V}$ to the Hochschild cochain complex $C^{\bullet}(\mathcal{A})$ of $\mathcal{A}$.

Let us briefly recall how to get the above data from an operad morphism

$$\mathsf{OC} \to \mathsf{End}_{(\mathcal{V},\mathcal{A})}\,. \tag{4.19}$$

[6] This fact was also observed in [6, Section 4.1].

The desired $\Lambda\mathsf{Lie}_\infty$ structure on $\mathcal{V}$

$$Q : \Lambda^2\mathsf{coCom}_\circ(\mathcal{V}) \to \mathcal{V}$$

comes from the action of corollas $\mathbf{t}^{\mathfrak{c}}_n$ on figure 4.1 for $n \geq 2$. Namely,

$$Q(v_1, \dots, v_n) = \mathbf{t}^{\mathfrak{c}}_n(v_1, \dots, v_n)\,, \qquad (4.20)$$

where $v_1, \dots, v_n \in \mathcal{V}$.

The desired $A_\infty$-structure

$$m : \Lambda\mathsf{coAs}_\circ(\mathcal{A}) \to \mathcal{A}$$

comes from the action of corollas $\mathbf{t}^{\mathfrak{o}}_k$ on figure 4.2 for $k \geq 2$. Namely,

$$m(a_1, \dots, a_k) = (-1)^{\varepsilon(a_1,\dots,a_k)}\mathbf{t}^{\mathfrak{o}}_k(a_1, \dots, a_k)\,, \qquad (4.21)$$

where $a_1, \dots, a_k \in \mathcal{A}$ and

$$\varepsilon(a_1, \dots, a_k) = |a_1|(k-1) + |a_2|(k-2) + \dots + |a_{k-1}|\,.$$

Finally the action of corollas $\mathbf{t}^{\mathfrak{o}}_{n,k}$ gives us the desired $\Lambda\mathsf{Lie}_\infty$-morphism from $\mathcal{V}$ to $C^\bullet(\mathcal{A})$

$$U : \Lambda^2\mathsf{coCom}(\mathcal{V}) \otimes T(\mathbf{s}^{-1}\mathcal{A}) \to \mathcal{A}\,.$$

Namely,

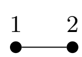
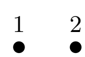
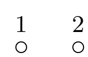
$$U(v_1, \dots, v_n; a_1, \dots, a_k) = (-1)^{\varepsilon'(v_1,\dots,v_n;a_1,\dots,a_k)}\mathbf{t}^{\mathfrak{o}}_{n,k}(v_1, \dots, v_n; a_1, \dots, a_k)\,, \qquad (4.22)$$

where

$$\varepsilon'(v_1, \dots, v_n; a_1, \dots, a_k) = k(|v_1| + \dots + |v_n|) + |a_1|(k-1) + |a_2|(k-2) + \dots + |a_{k-1}|\,. \qquad (4.23)$$

## 5 Stable formality quasi-isomorphisms and their homotopies

Several vectors of $\mathsf{KGra}$ will play a special role in the definition and of a stable formality quasi-isomorphism and in further considerations. For this reason, we will reserve separate symbols for these vectors and use these symbols in this paper and in paper [7]. These are

$\Gamma_{\bullet\!-\!\bullet} =$  $\Gamma_{\bullet\bullet} =$  $\Gamma_{\circ\circ} =$  (5.1)

and the series of "brooms" for $k \geq 0$ depicted on figure 5.1.

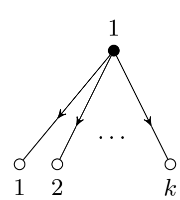
$\Gamma^{\mathrm{br}}_k \quad =$ 


Fig. 5.1: Edges are ordered in this way $(1_{\mathfrak{c}}, 1_{\mathfrak{o}}) < (1_{\mathfrak{c}}, 2_{\mathfrak{o}}) < \dots < (1_{\mathfrak{c}}, k_{\mathfrak{o}})$

Note that the graph $\Gamma^{\rm br}_0 \in \mathsf{KGra}(1,0)^{\mathfrak{o}}$ consists of a single black vertex labeled by 1 and it has no edges.

According to Section 3.5, the 2-colored operad $\mathsf{KGra}$ acts on the pair $(V_A, A)$ where $A$ (resp. $V_A$) is the algebra of functions (resp. the algebra of polyvector fields) on a graded affine space. Hence, every morphism of operads (in $\mathsf{Ch}_{\mathbb{K}}$) $F : \mathsf{OC} \to \mathsf{KGra}$ gives us a $\Lambda\mathsf{Lie}_\infty$-structure on $V_A$, an $A_\infty$-structure on $A$ and an $\Lambda\mathsf{Lie}_\infty$-morphism from $V_A$ to the Hochschild cochain complex $C^\bullet(A)$ of $A$. This observation motivates the following definition.

**Definition 5.1** *A stable formality quasi-isomorphism is a morphism of 2-colored operads in the category of cochain complexes*

$$F : \mathsf{OC} \to \mathsf{KGra} \tag{5.2}$$

*satisfying the following "boundary conditions":*

$$F(\mathbf{t}^{\mathfrak{c}}_n) = \begin{cases} \Gamma_{\bullet\!-\!\bullet} & \text{if} \quad n = 2\,, \\ 0 & \text{if} \quad n \ge 3\,, \end{cases} \tag{5.3}$$

$$F(\mathbf{t}^{\mathfrak{o}}_2) = \Gamma_{\circ\,\circ}\,, \tag{5.4}$$

*and*

$$F(\mathbf{t}^{\mathfrak{o}}_{1,k}) = \Gamma^{\rm br}_k\,, \tag{5.5}$$

*where $\mathbf{t}^{\mathfrak{c}}_n$, $\mathbf{t}^{\mathfrak{o}}_k$, and $\mathbf{t}^{\mathfrak{o}}_{n,k}$ are corollas depicted on figures 4.1, 4.2, 4.3, respectively, and $\Gamma_{\bullet\!-\!\bullet}$, $\Gamma_{\circ\,\circ}$ and $\Gamma^{\rm br}_k$ are the vectors of $\mathsf{KGra}$ specified in the beginning of this section.*

To interpret the "boundary conditions" we consider the OCHA structure induced by the morphism $F$ (5.2) on the pair $(V_A, A)$.

The first condition (eq. (5.3)) implies that the $\Lambda\mathsf{Lie}_\infty$-structure on polyvector fields induced by the morphism $F$ coincides with the standard Schouten-Nijenhuis algebra structure.

The second condition (eq. (5.4)) implies that the binary operation of the induced $A_\infty$-structure on $A$ coincides with the ordinary (commutative) multiplication. For degree reasons, the image $F(\mathbf{t}^{\mathfrak{o}}_k)$ of the corolla $\mathbf{t}^{\mathfrak{o}}_k$ in $\mathsf{KGra}(0,k)^{\mathfrak{o}}$ is zero for all $k \ge 3$. Thus the induced $A_\infty$-structure on $A$ coincides with the original associative (and commutative) algebra structure.

The third boundary condition (eq. (5.5)) implies that the corresponding $\Lambda\mathsf{Lie}_\infty$-morphism from $V_A$ to $C^\bullet(A)$ starts with the Hochschild-Kostant-Rosenberg embedding. The latter condition guarantees that the induced $\Lambda\mathsf{Lie}_\infty$-morphism is a quasi-isomorphism.

## 5.1 Stable formality quasi-isomorphisms as MC elements. Homotopies of stable formality quasi-isomorphisms

Due to Proposition 2.2 and Remark 4.1 stable formality quasi-isomorphisms are in bijection with MC elements $\alpha$ of the Lie algebra

$$\mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra}) \tag{5.6}$$

subject to the three conditions

$$\alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{c}}_n) = \begin{cases} \Gamma_{\bullet\!-\!\bullet} & \text{if} \quad n = 2\,, \\ 0 & \text{if} \quad n \ge 3\,, \end{cases} \tag{5.7}$$

$$\alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_2) = \Gamma_{\circ\circ}\,, \tag{5.8}$$

and

$$\alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{1,k}) = \Gamma^{\mathrm{br}}_k\,, \tag{5.9}$$

where $\mathbf{t}^{\mathfrak{c}}_n$, $\mathbf{t}^{\mathfrak{o}}_k$, and $\mathbf{t}^{\mathfrak{o}}_{n,k}$ are corollas depicted on figures 4.1, 4.2, 4.3, respectively, and $\Gamma_{\bullet\!-\!\bullet}$, $\Gamma_{\circ\circ}$ and $\Gamma^{\mathrm{br}}_k$ are the vectors of $\mathsf{KGra}$ specified in the beginning of this section.

We would like to remark that, since all vectors in $\mathsf{KGra}(0,k)^{\mathfrak{o}}$ have degree zero, we have

$$\alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_k) = 0\,, \tag{5.10}$$

for all $k \ge 3$ and for all degree 1 elements $\alpha$ in (5.6).

In what follows we denote by $\alpha_F$ the MC element in (5.6) corresponding a stable formality quasi-isomorphism $F$.

According to Section 2.3 the Lie algebra $\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$ is equipped with the "arity" filtration $\mathcal{F}_{\bullet}\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$ such that $\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$ is complete with respect to this filtration.

Hence, following general theory from Appendix C, the set of MC elements of the Lie algebra (5.6) is equipped with the action of pro-unipotent group

$$\exp\Big(\mathcal{F}_1\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})^0\Big)\,. \tag{5.11}$$

We claim that

**Proposition 5.1** *If a degree zero vector*

$$\xi \in \mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra}) \tag{5.12}$$

*satisfies the boundary conditions*

$$\xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{c}}_n) = 0 \qquad \forall \;\; n \ge 2 \tag{5.13}$$

$$\xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{1,k}) = 0 \qquad \forall \;\; k \ge 0 \tag{5.14}$$

*then*

$$\xi \in \mathcal{F}_1\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})\,. \tag{5.15}$$

*The subspace of degree zero vectors $\xi$ (5.12) satisfying conditions (5.13) and (5.14) form a Lie subalgebra of $\mathcal{F}_1\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})^0$. Finally, if $\alpha$ is a MC element of the Lie algebra $\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$ satisfying the boundary conditions (5.7), (5.8), (5.9) and $\xi$ is a degree zero vector (5.12) satisfying (5.13) and (5.14) then the MC element*

$$\alpha' = \exp(\xi)\,(\alpha) \tag{5.16}$$

*also satisfies conditions (5.7), (5.8), (5.9).*

Proof. Inclusion (5.15) follows immediately from equation (5.14) for $k = 0$.

Next, since all vectors in $\mathsf{KGra}(0,k)^{\mathfrak{o}}$ have degree zero, we conclude that

$$\xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_k) = 0 \qquad \forall \;\; k \ge 2 \tag{5.17}$$

for any degree zero vector (5.12).

Using equation (5.17) it is easy to see that

$$[\xi_1, \xi_2](\mathbf{s}^{-1}\, \mathbf{t}^{\mathfrak{o}}_{1,k}) = 0$$

for all degree zero vectors $\xi_1, \xi_2 \in \mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra})$.

Moreover, the vector $[\xi_1, \xi_2]$ satisfies condition (5.13) if so do both $\xi_1$ and $\xi_2$. Thus, vectors (5.12) satisfying conditions (5.13) and (5.14) indeed form a Lie subalgebra of $\mathcal{F}_1\mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra})^0$.

Finally, using (5.13), (5.14), and (5.17), it is easy to see that $\alpha'$ in (5.16) satisfies conditions (5.7), (5.8), (5.9) if so does $\alpha$.

□

We can now give the definition of homotopy between two stable formality quasi-isomorphisms.

**Definition 5.2** *We say that a stable formality quasi-isomorphism $F$ (5.2) is homotopy equivalent to $\widetilde{F}$ if the corresponding MC elements*

$$\alpha_F, \alpha_{\widetilde{F}} \in \mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra})$$

*are isomorphic via $\exp(\xi)$, where $\xi$ is a degree zero element in $\mathcal{F}_1\mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra})$ satisfying the "boundary conditions" (5.13) and (5.14).*

**Remark 5.1** Proposition 5.1 implies that the defined relation on the set of stable formality quasi-isomorphisms is indeed an equivalence relation.

To explain our motivation behind Definition 5.2 we consider the pair $(V_A, A)$, where $A$ is a finitely generated commutative algebra in $\mathsf{grVect}_{\mathbb{K}}$ and $V_A$ be the algebra of polyvector fields on the corresponding (graded) affine space.

Recall that a stable formality quasi-isomorphism $F$ gives us a $\Lambda\mathsf{Lie}_\infty$ quasi-isomorphism $U_F$ from $V_A$ to $C^\bullet(A)$ which admits graphical expansion.

We claim that, if two stable formality quasi-isomorphisms $F$ and $\widetilde{F}$ are homotopy equivalent, then the corresponding $\Lambda\mathsf{Lie}_\infty$-morphisms $U_F$ and $U_{\widetilde{F}}$ are also homotopy equivalent. Furthermore, the homotopy between $U_F$ and $U_{\widetilde{F}}$ admits graphical expansion.

Indeed, according to [4], any $\Lambda\mathsf{Lie}_\infty$-morphism

$$U : V_A \rightsquigarrow C^\bullet(A)$$

is a MC element in the following auxiliary Lie algebra

$$\mathbf{s}\mathrm{Hom}(\mathbf{s}^2 S(\mathbf{s}^{-2} V_A), C^\bullet(A))\,. \tag{5.18}$$

(The differential and the Lie bracket on (5.18) are expressed naturally in terms of the differentials and the Lie brackets on $V_A$ and $C^\bullet(A)$.)

Furthermore, two $\Lambda\mathsf{Lie}_\infty$-morphisms

$$U : V_A \rightsquigarrow C^\bullet(A) \qquad \text{and} \qquad \widetilde{U} : V_A \rightsquigarrow C^\bullet(A)$$

are homotopy equivalent if and only if the corresponding MC elements in the Lie algebra (5.18) are isomorphic.

By merely unfolding definitions it is not hard to see that, if MC elements

$$\alpha_F, \alpha_{\widetilde{F}} \in \mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra})$$

are isomorphic via $\exp(\xi)$ for a degree zero vector $\xi \in \mathcal{F}_1 \mathrm{Conv}(\mathfrak{oc}_\circ^\vee, \mathsf{KGra})$ satisfying (5.13) and (5.14), then the MC elements in (5.18) corresponding to $U_F$ and $U_{\widetilde{F}}$ are isomorphic via

$$\exp(\xi')$$

where $\xi'$ is the degree zero vector in (5.18) given by the formula:

$$\xi'(v_1, v_2, \dots, v_n; a_1, a_2, \dots, a_n) = \tag{5.19}$$

$$(-1)^{\varepsilon'(v_1,\dots,v_n;a_1,\dots,a_k)} \xi(\mathbf{s}^{-1}\mathfrak{t}^{\mathfrak{o}}_{n,k})(v_1, v_2, \dots, v_n; a_1, a_2, \dots, a_n)\,,$$

where $\varepsilon'(v_1, \dots, v_n; a_1, \dots, a_k)$ is defined in (4.23).

**Example 5.2** In his famous paper [24] M. Kontsevich proposed a construction of a $\Lambda\mathsf{Lie}_\infty$ quasi-isomorphism from the Lie algebra of polyvector fields $V_A$ on $\mathbb{R}^d$ to polydifferential operators on $\mathbb{R}^d$. The structure maps of this $\Lambda\mathsf{Lie}_\infty$ quasi-isomorphism are defined using graphical expansion and the $\Lambda\mathsf{Lie}_\infty$ quasi-isomorphism starts with the standard Hochschild-Kostant-Rosenberg embedding. Thus Kontsevich's construction from [24] gives us a stable formality quasi-isomorphism over any extension of the field $\mathbb{R}$.

# 6 The action of Kontsevich's graph complex on stable formality quasi-isomorphisms

It is possible to produce new homotopy types of stable formality quasi-isomorphisms using the action of Kontsevich's graph complex on the Lie algebra

$$\mathrm{Conv}(\mathfrak{oc}_\circ^\vee, \mathsf{KGra})\,.$$

We describe this action here.

## 6.1 Reminder of the full graph complex dfGC

Let us consider the Lie algebra (in $\mathsf{grVect}_{\mathbb{K}}$)

$$\mathsf{dfGC} = \mathrm{Conv}(\Lambda^2\mathsf{coCom}, \mathsf{dGra})\,. \tag{6.1}$$

Since

$$\mathrm{Conv}(\Lambda^2\mathsf{coCom}, \mathsf{dGra}) = \prod_{n=1}^{\infty} \mathbf{s}^{2n-2}\big(\mathsf{dGra}(n)\big)^{S_n}$$

vectors in (6.1) are (possibly infinite) linear combinations

$$\gamma = \sum_{n=1}^{\infty} \gamma_n$$

where $\gamma_n$ is an $S_n$-invariant vector in $\mathsf{dGra}(n)$.

If all graphs in the linear combination $\gamma_n \in (\mathsf{dGra}(n))^{S_n}$ have the same number of edges $e$ then $\gamma_n$ is a homogeneous vector in $\mathsf{dfGC}$ of degree

$$|\gamma_n| = 2n - 2 - e\,. \tag{6.2}$$

For example, the vector $\Gamma_{\bullet\!\to\!\bullet} \in \mathsf{dGra}(2)$ defined in (5.1) is $S_2$-invariant and hence is a vector in $\mathsf{dfGC}$. According to (6.2) the vector $\Gamma_{\bullet\!\to\!\bullet}$ carries degree $1$. A direct computation shows that

$$[\Gamma_{\bullet\!\to\!\bullet}, \Gamma_{\bullet\!\to\!\bullet}] = 0\,. \tag{6.3}$$

Hence, $\Gamma_{\bullet\!\to\!\bullet}$ is a MC element and it can be used to equip the graded vector space (6.1) with the non-zero differential

$$\partial = \mathrm{ad}_{\Gamma_{\bullet\!\to\!\bullet}}\,. \tag{6.4}$$

**Definition 6.1** *The graded vector space* $\mathsf{dfGC}$ (6.1) *with the differential* (6.4) *is called the full graph complex.*

For example the graph $\Gamma_{\bullet} \in \mathrm{dgra}_1$ which consists of a single vertex without edges gives us a degree zero vector $\mathsf{dfGC}$. According to the definition of the Lie bracket on $\mathsf{dfGC}$ we have

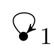
$$[\Gamma_{\bullet\!\to\!\bullet}, \Gamma_{\bullet}] = \Gamma_{\bullet\!\to\!\bullet} \circ_1 \Gamma_{\bullet} + \sigma_{12}(\Gamma_{\bullet\!\to\!\bullet} \circ_1 \Gamma_{\bullet}) - \Gamma_{\bullet} \circ_1 \Gamma_{\bullet\!\to\!\bullet} = \Gamma_{\bullet\!\to\!\bullet} + \Gamma_{\bullet\!\to\!\bullet} - \Gamma_{\bullet\!\to\!\bullet} = \Gamma_{\bullet\!\to\!\bullet}\,, \tag{6.5}$$

where $\sigma_{12}$ is the transposition in $S_2$.

Thus $\Gamma_{\bullet}$ is not a cocycle in $\mathsf{dfGC}$.

Let us also consider the graph $\Gamma_{\circlearrowleft} \in \mathrm{dgra}_1$ which consists of the single loop.

$$\Gamma_{\circlearrowleft} = \tag{6.6}$$




The computation on figure 6.1 proves that $\Gamma_{\circlearrowleft}$ is a cocycle in $\mathsf{dfGC}$. It is obvious that the cocycle $\Gamma_{\circlearrowleft}$ is non-trivial and it has degree $-1$.

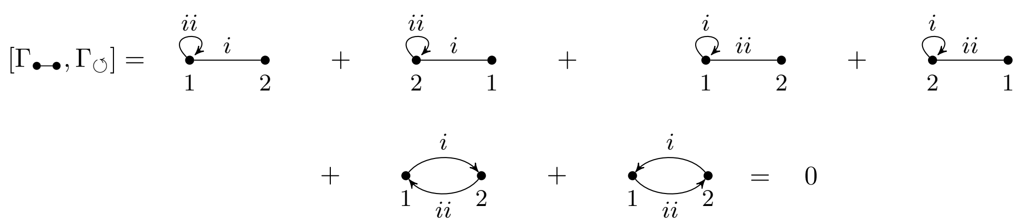



Fig. 6.1: Roman numerals are used to specify the total orders on the sets of edges. For example, in the first graph we have $(1,2) < (1,1)$

According to Section 2.3 the Lie algebra $\mathsf{dfGC}$ is equipped with the descending filtration (2.45) such that $\mathsf{dfGC}$ is complete with respect to this filtration. Since $\Gamma_{\bullet\!\to\!\bullet} \in \mathcal{F}_1(\mathsf{dfGC})$ the differential (6.4) is compatible with the filtration on $\mathsf{dfGC}$.

Since we have exactly two graphs $\Gamma_{\bullet}$ and $\Gamma_{\circlearrowleft}$ in $\mathrm{dgra}_1$, $\Gamma_{\bullet}$ is not a cocycle in $\mathsf{dfGC}$ and the cocycle $\Gamma_{\circlearrowleft}$ has degree $-1$ we conclude that each degree zero cocycle $\gamma \in \mathsf{dfGC}$ has the property

$$\gamma \in \mathcal{F}_1\mathsf{dfGC}\,. \tag{6.7}$$

Hence, the Lie algebra $H^0(\mathsf{dfGC})$ is pro-nilpotent.

To give an example of a degree zero cocycle in $\mathsf{dfGC}$ we consider the tetrahedron in $\mathsf{dGra}(4)$ depicted on figure 6.2. This graph is invariant with respect to the action of $S_4$ and hence it can be viewed as vector in $\mathsf{dfGC}$. According to (6.2) this vector has degree zero. A direct computation shows that it is a cocycle and it is easy to see that this cocycle is non-trivial.

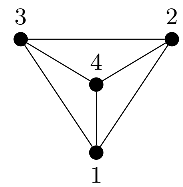



Fig. 6.2: We may choose this order on the set of edges: $(1,2) < (1,3) < (1,4) < (2,3) < (2,4) < (3,4)$

In fact it was proved in [37, Proposition 15] that for every odd number $n \geq 3$ there exists a non-trivial cocycle which has a non-zero coefficient in front of the wheel with $n$ spokes (see figure 6.3). Note that, labels on vertices do not play an important role because vectors in $\mathsf{dfGC}$ are invariant under the action of the symmetric group.

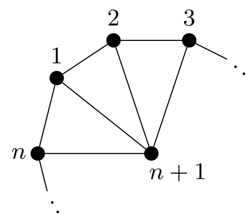



Fig. 6.3: Here $n$ is an odd integer $\geq 3$

Using the results [37] of T. Willwacher, one can prove[7] the following statement

**Theorem 6.1 (T. Willwacher, [37])** *For the full graph complex* $\mathsf{dfGC}$ *we have*

$$H^q(\mathsf{dfGC}) = \mathbf{0} \qquad \forall \;\; q < -1\,. \tag{6.8}$$

*The vector space* $H^{-1}(\mathsf{dfGC})$ *is one-dimensional and it is spanned by the class of the cocycle* (6.6)*. Finally, the pro-nilpotent Lie algebras* $H^0(\mathsf{dfGC})$ *and* $\mathfrak{grt}$ *are isomorphic.*

**Remark 6.1** Let $\Gamma$ be an element in $\mathrm{dgra}_n$. We say that a vertex $v$ of $\Gamma$ is a *pike* if $v$ has valency 1 and the edge adjacent to $v$ terminates at $v$. We observe that, due Proposition 27 from [37], any cocycle $\gamma$ in $\mathsf{dfGC}$ is cohomologous to a cocycle in which all graphs do not have pikes.

### 6.2 The action of dfGC on stable formality quasi-isomorphisms

For our purposes it is convenient to extend the Lie algebra $\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$ (5.6) to the Lie algebra

$$\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra})\,, \tag{6.9}$$

where $\mathfrak{oc}^{\vee}$ is viewed as a pseudo-cooperad via the forgetful functor.

The natural surjective map of collections

$$\mathfrak{oc}^{\vee} \twoheadrightarrow \mathfrak{oc}^{\vee}_{\circ}$$

[7] For a detailed proof of Theorem 6.1 we refer the reader to paper [9].

gives us the inclusion of the Lie algebra $\mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra})$ into $\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$

$$\mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra}) \hookrightarrow \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})\,. \tag{6.10}$$

Thus MC elements of the Lie algebra $\mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra})$ can be viewed as MC elements of its extension $\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$.

Using equation (4.17) we define a natural embedding of $\mathsf{dfGC}$ (6.1) into the Lie algebra $\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$

$$J : \mathsf{dfGC} \hookrightarrow \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})\,. \tag{6.11}$$

This embedding is given by the formulas

$$J(\gamma)\Big|_{\mathfrak{oc}^\vee(n,0)^{\mathfrak{c}}} = \gamma\,, \qquad J(\gamma)\Big|_{\mathfrak{oc}^\vee(n,k)^{\mathfrak{o}}} = 0\,. \tag{6.12}$$

The embedding $J$ is obviously compatible with the Lie brackets and with the filtrations by arity on $\mathsf{dfGC}$ and $\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$. However, we should point out that $J$ is not compatible with the differentials. Indeed, the Lie algebra $\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$ carries the zero differential while $\mathsf{dfGC}$ carries the non-zero differential (6.4).

Let $\alpha_F$ be a MC element in $\mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra})$ corresponding to a stable formality quasi-isomorphism (5.2). We claim that

**Proposition 6.1** *For every degree zero cocycle $\gamma \in \mathsf{dfGC}$ the equation*

$$\alpha' = \exp(\mathrm{ad}_{J(\gamma)})\alpha_F \tag{6.13}$$

*defines a MC element $\alpha'$ in* $\mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra})$ *satisfying conditions* (5.7), (5.8), *and* (5.9).

Proof. It is obvious that $\alpha'$ satisfies the MC equation in $\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$.

Furthermore, since each degree zero cocycle in $\mathsf{dfGC}$ belongs to $\mathcal{F}_1\mathsf{dfGC}$, the MC element $\alpha'$ belongs to the Lie subalgebra

$$\mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra}) \subset \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}),\,.$$

Next, using the cocycle condition for $\gamma$

$$[\Gamma_{\bullet\!-\!\bullet}, \gamma] = 0$$

it is not hard to show that $\alpha'$ satisfies condition (5.7).

Finally, it is straightforward to verify that $\alpha'$ also satisfies (5.8) and (5.9).

$\square$

Due to Proposition 6.1 degree zero cocycles in $\mathsf{dfGC}$ act on stable formality quasi-isomorphisms. In the following proposition we list two important properties of this action.

**Proposition 6.2** *Let $\gamma$ be a degree zero cocycle in* $\mathsf{dfGC}$. *If $\alpha$ and $\widetilde{\alpha}$ are MC elements of the Lie algebra* (5.6) *corresponding to homotopy equivalent stable formality quasi-isomorphisms $F$ and $\widetilde{F}$, then the MC elements*

$$\exp(\mathrm{ad}_{J(\gamma)})\alpha\,, \quad \textit{and} \quad \exp(\mathrm{ad}_{J(\gamma)})\widetilde{\alpha}$$

*correspond to homotopy equivalent stable formality quasi-isomorphisms.*

*Furthermore, if*

$$\gamma = [\Gamma_{\bullet\!-\!\bullet}, \psi] \tag{6.14}$$

*then there exists a degree zero vector*

$$\xi \in \mathcal{F}_1 \mathrm{Conv}(\mathfrak{oc}_\circ^\vee, \mathsf{KGra})$$

*which satisfies* (5.13) *and such that*

$$\exp(\mathrm{ad}_{J(\gamma)})\alpha = \exp(\mathrm{ad}_\xi)\alpha\,. \tag{6.15}$$

*If, in addition,*

$$\psi \in \mathcal{F}_{n-1}\mathsf{dfGC}$$

*then the vector* $\xi$ *in* (5.13) *can be chosen in such a way that*

$$\xi \in \mathcal{F}^{\mathfrak{c}}_{n-1} \mathrm{Conv}(\mathfrak{oc}_\circ^\vee, \mathsf{KGra}) \tag{6.16}$$

*and*

$$\xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,0}) = \psi(1_n)\,, \tag{6.17}$$

*where* $1_n$ *is the generator*

$$\mathbf{s}^{2-2n}1 \in \mathbf{s}^{2-2n}\mathbb{K} \cong \Lambda^2\mathsf{coCom}(n)\,.$$

Proof. Since $\alpha$ and $\widetilde{\alpha}$ represent homotopy equivalent stable formality quasi-isomorphisms, there exists a degree zero vector

$$\xi \in \mathcal{F}_1 \mathrm{Conv}(\mathfrak{oc}_\circ^\vee, \mathsf{KGra})$$

which satisfies (5.13) and such that

$$\widetilde{\alpha} = \exp(\mathrm{ad}_\xi)\alpha\,. \tag{6.18}$$

Applying $\exp(\mathrm{ad}_{J(\gamma)})$ to both sides of equation (6.18) we get

$$\exp(\mathrm{ad}_{J(\gamma)})\widetilde{\alpha} = \exp(\mathrm{ad}_{J(\gamma)})\exp(\mathrm{ad}_\xi)\alpha = \tag{6.19}$$
$$\exp(\mathrm{ad}_{\widetilde{\xi}})\big(\exp(\mathrm{ad}_{J(\gamma)})\alpha\big)\,,$$

where

$$\widetilde{\xi} = \exp(\mathrm{ad}_{J(\gamma)})\,\xi \tag{6.20}$$

The vector $\widetilde{\xi}$ obviously belongs to $\mathcal{F}_1 \mathrm{Conv}(\mathfrak{oc}_\circ^\vee, \mathsf{KGra})$. Furthermore, $\widetilde{\xi}$ satisfies the condition

$$\widetilde{\xi}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{c}}_n) = 0\,, \qquad \forall \;\; n \ge 2$$

since so does $\xi$.

Thus the MC elements

$$\exp(\mathrm{ad}_{J(\gamma)})\alpha\,, \quad \text{and} \quad \exp(\mathrm{ad}_{J(\gamma)})\widetilde{\alpha}$$

indeed correspond to homotopy equivalent stable formality quasi-isomorphisms.

To prove the second statement we introduce the Lie algebra (in $\mathsf{grVect}_{\mathbb{K}}$)

$$\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})\{t\}\,. \tag{6.21}$$

A vector in $\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})\{t\}$ is a formal Taylor power series in an auxiliary (degree zero) variable $t$

$$\sum_{r=0}^{\infty} t^r f_r \in \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})[[t]] \tag{6.22}$$

satisfying the property

$$\exists \;\; m \ge 0 \quad \text{such that} \quad f_r \in \mathcal{F}_{m+r}\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})\,. \tag{6.23}$$

Let us denote by $\alpha(t)$ the following vector in (6.21)

$$\alpha(t) = \exp(t\, \mathrm{ad}_{J(\gamma)})\alpha\,. \tag{6.24}$$

It is easy to see that $\alpha(t)$ enjoys the MC equation

$$[\alpha(t), \alpha(t)] = 0 \tag{6.25}$$

and the condition

$$\alpha(t)\,(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{c}}_n) = \begin{cases} \Gamma_{\bullet\!-\!\bullet} & \text{if} \quad n = 2\,, \\ 0 & \text{if} \quad n \ge 3\,, \end{cases} \tag{6.26}$$

since so does $\alpha$ and since $\gamma$ is cocycle in $\mathsf{dfGC}$. Furthermore, $\alpha(t)$ satisfies the following (formal) differential equation

$$\frac{d}{dt}\,\alpha(t) = [J(\gamma), \alpha(t)] \tag{6.27}$$

with the initial condition

$$\alpha(t)\Big|_{t=0} = \alpha\,. \tag{6.28}$$

Let us now assume that

$$\gamma = [\Gamma_{\bullet\!-\!\bullet}, \psi] \tag{6.29}$$

for a degree $-1$ vector in $\mathsf{dfGC}$.

The degree $-1$ component of the quotient

$$\mathsf{dfGC}\Big/\mathcal{F}_1\mathsf{dfGC}$$

is spanned by the graph $\Gamma_{\circlearrowleft}$ (6.6) which is a cocycle due to the computation shown on figure 6.1. Thus we may assume, without loss of generality that

$$\psi \in \mathcal{F}_1\mathsf{dfGC}\,. \tag{6.30}$$

Since the map $J$ (6.11) is compatible with Lie brackets, we have

$$J(\gamma) = [J(\Gamma_{\bullet\!-\!\bullet}), J(\psi)]$$

and hence the vector $\alpha(t)$ satisfies the equation

$$\frac{d}{dt}\,\alpha(t) = \big[[J(\Gamma_{\bullet\!-\!\bullet}), J(\psi)], \alpha(t)\big]\,. \tag{6.31}$$

Let us denote by $\Delta\alpha$ the difference

$$\Delta\alpha = \alpha(t) - J(\Gamma_{\bullet\!-\!\bullet}) \in \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})\{t\} \tag{6.32}$$

The MC equation (6.25) for $\alpha(t)$ implies that

$$[J(\Gamma_{\bullet\!-\!\bullet}), \Delta\alpha] + \frac{1}{2}[\Delta\alpha, \Delta\alpha] = 0\,. \tag{6.33}$$

Furthermore, due to equation (6.26), we have

$$\Delta\alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{c}}_n) = 0 \qquad \forall\;\; n \ge 2\,. \tag{6.34}$$

Using the Jacobi idenity, identity $[J(\Gamma_{\bullet\!-\!\bullet}), J(\Gamma_{\bullet\!-\!\bullet})] = 0$, and equation (6.33) we rewrite equation (6.31) as follows

$$\frac{d}{dt}\,\alpha(t) = \big[[J(\Gamma_{\bullet\!-\!\bullet}), J(\psi)], J(\Gamma_{\bullet\!-\!\bullet})\big] + \big[[J(\Gamma_{\bullet\!-\!\bullet}), J(\psi)], \Delta\alpha\big] = \big[[J(\Gamma_{\bullet\!-\!\bullet}), J(\psi)], \Delta\alpha\big] =$$
$$-\big[[J(\psi), \Delta\alpha], J(\Gamma_{\bullet\!-\!\bullet})\big] - \big[[J(\Gamma_{\bullet\!-\!\bullet}), \Delta\alpha], J(\psi)\big] =$$
$$-\big[[J(\psi), \Delta\alpha], J(\Gamma_{\bullet\!-\!\bullet})\big] + \frac{1}{2}\big[[\Delta\alpha, \Delta\alpha], J(\psi)\big] =$$
$$-\big[[J(\psi), \Delta\alpha], J(\Gamma_{\bullet\!-\!\bullet})\big] - \frac{1}{2}\big[[\Delta\alpha, J(\psi)], \Delta\alpha\big] - \frac{1}{2}\big[[J(\psi), \Delta\alpha], \Delta\alpha\big] =$$
$$-\big[[J(\psi), \Delta\alpha], \alpha(t)\big]\,.$$

Thus the vector $\alpha(t)$ (6.24) satisfies the (formal) differential equation

$$\frac{d}{dt}\,\alpha(t) = [\xi(t), \alpha(t)]\,, \tag{6.35}$$

$\xi(t)$ is the degree zero vector

$$\xi(t) = -[J(\psi), \Delta\alpha] \in \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})\{t\}\,. \tag{6.36}$$

Inclusion (6.30) implies that the zeroth term in the expansion of $\xi$ belongs to $\mathcal{F}_1\mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra})$. Moreover, due to equation (6.34), we have

$$\xi(t)\big(\mathbf{t}^{\mathfrak{c}}_n\big) = 0 \qquad \forall\;\; n \ge 2\,.$$

Thus, Theorem C.1 from Appendix C.1 implies the existence of the desired vector

$$\xi \in \mathcal{F}_1\mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra})$$

for which equation (6.15) holds.

If

$$\psi \in \mathcal{F}_{n-1}\mathsf{dfGC}$$

then $\xi(t) \in \mathcal{F}_{n-1}\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})\{t\}$ and hence

$$\xi \in \mathcal{F}_{n-1}\mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra})\,.$$

To compute $\xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,0})$ we notice that

$$\xi - \xi\Big|_{t=0} \in \mathcal{F}_n\mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra})\,.$$

Therefore,

$$\xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,0}) = -[J(\psi), \alpha - \Gamma_{\bullet\!\!-\!\!\bullet}](\mathbf{t}^{\mathfrak{o}}_{n,0}) =$$
$$= \alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{1,0}) \circ_{1,\mathfrak{c}} J(\psi)(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{c}}_{n}) =$$
$$\Gamma^{\mathrm{br}}_{0} \circ_{1,\mathfrak{c}} \psi(1_n) = \psi(1_n)\,.$$

The desired statements are proved. □

Proposition 6.2 implies that

**Corollary 6.1** *The action of degree zero cocycles of* $\mathsf{dfGC}$ *on stable formality quasi-isomorphisms descends to an action of* $\exp\big(H^0(\mathsf{dfGC})\big)$ *on homotopy classes.*

Proof. Let $\gamma$ be a degree zero cocycle of $\mathsf{dfGC}$. Due to the first statement of Proposition 6.2, $\exp(\gamma)$ transforms homotopy equivalent stable formality quasi-isomorphisms to homotopy equivalent stable formality quasi-isomorphisms.

Thus it remains to prove that, if $\gamma'$ is cohomologous to $\gamma$ then MC elements

$$\exp(\mathrm{ad}_{J(\gamma')})\alpha \qquad \text{and} \qquad \exp(\mathrm{ad}_{J(\gamma)})\alpha \tag{6.37}$$

are connected by the action of $\exp(\mathrm{ad}_\xi)$ for a vector

$$\xi \in \mathcal{F}_1 \mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra})$$

satisfying condition (5.13)

Using the fact that the difference $\gamma' - \gamma$ is exact, it is easy to see that

$$\mathrm{CH}(-\gamma, \gamma')$$

is also exact.

Therefore, due to the second statement of Proposition 6.2, the MC elements

$$\exp(\mathrm{ad}_{J(\gamma)})\,\exp(\mathrm{ad}_{J(\gamma')})\,\alpha \qquad \text{and} \qquad \alpha$$

are connected by the action of $\exp(\mathrm{ad}_\xi)$ for a vector $\xi \in \mathcal{F}_1 \mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra})$ satisfying condition (5.13).

Hence, the MC elements (6.37) represent homotopy equivalent stable formality quasi-isomorphisms. □

**Remark 6.2** Let $A$ be a finitely generated commutative algebra in $\mathsf{grVect}_{\mathbb{K}}$ and $V_A$ be the algebra of polyvector fields on the corresponding (graded) affine space. It is not hard to see that $\Lambda\mathsf{Lie}_\infty$-morphisms from $V_A$ to $C^\bullet(A)$ which correspond to $\alpha_F$ and (6.13) are connected by the action described in [23, Section 5] by M. Kontsevich.

Let us now state the main result of this paper

**Theorem 6.2** *The pro-unipotent group* $\exp(H^0(\mathsf{dfGC}))$ *acts simply transitively on the set of homotopy classes of stable formality quasi-isomorphisms.*

The proof of this theorem occupies the next two sections of the paper and it depends on a few technical statements which are proved Appendices A and B.

To conclude this section, we observe that the combination of Theorems 6.1 and 6.2 implies that

**Corollary 6.2** *The set of homotopy classes of stable formality quasi-isomorphisms is a torsor for the Grothendieck-Teichmüller group* $\mathsf{GRT}$. $\square$

In subsequent paper [7] we develop Tamarkin's construction in stable setting and prove that this construction gives us an isomorphism from the $\mathsf{GRT}$-torsor of Drinfeld's associators to the $\mathsf{GRT}$-torsor of stable formality quasi-isomorphisms.

# 7 The action of $\exp\big(H^0(\mathsf{dfGC})\big)$ is transitive

Let $F$ and $\widetilde{F}$ be operad morphisms

$$F, \widetilde{F} \;:\; \mathsf{OC} \to \mathsf{KGra}$$

both satisfying boundary conditions (5.3), (5.4), and (5.5). Furthermore, let $\alpha$ and $\widetilde{\alpha}$ be MC elements in $\mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra})$ corresponding to $F$ and $\widetilde{F}$, respectively. In other words,

$$\alpha\,\mathbf{s}^{-1} \;=\; F\Big|_{\mathfrak{oc}}\,, \tag{7.1}$$

and

$$\widetilde{\alpha}\,\mathbf{s}^{-1} \;=\; \widetilde{F}\Big|_{\mathfrak{oc}}\,, \tag{7.2}$$

Since $\alpha$ and $\widetilde{\alpha}$ satisfy the equations

$$[\alpha, \alpha] = 0\,, \qquad [\,\widetilde{\alpha}\,, \,\widetilde{\alpha}\,] = 0 \tag{7.3}$$

the difference

$$\delta\alpha = \widetilde{\alpha} - \alpha \tag{7.4}$$

is a MC element in the Lie algebra $\mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra})$ with the new differential $\mathrm{ad}_\alpha$. In other words,

$$[\alpha, \delta\alpha] + \frac{1}{2}[\delta\alpha, \delta\alpha] = 0\,. \tag{7.5}$$

Since $\mathfrak{oc}^\vee_\circ$ is spanned by vectors $\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{c}}_m$ $(m \ge 2)$, $\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_k$ $(k \ge 2)$ and $\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k}$ $(m \ge 1, k \ge 0)$, equation (7.5) is equivalent to the following three identities (here we also use (2.56))

$$\mu\big(\alpha\mathbf{s}^{-1} \otimes \delta\alpha\mathbf{s}^{-1}\,(\mathcal{D}(\mathbf{t}^{\mathfrak{c}}_m)) + \delta\alpha\mathbf{s}^{-1} \otimes \alpha\mathbf{s}^{-1}\,(\mathcal{D}(\mathbf{t}^{\mathfrak{c}}_m)) + \delta\alpha\mathbf{s}^{-1} \otimes \delta\alpha\mathbf{s}^{-1}\,(\mathcal{D}(\mathbf{t}^{\mathfrak{c}}_m))\big) = 0\,, \tag{7.6}$$

$$\mu\big(\alpha\mathbf{s}^{-1} \otimes \delta\alpha\mathbf{s}^{-1}\,(\mathcal{D}(\mathbf{t}^{\mathfrak{o}}_k)) + \delta\alpha\mathbf{s}^{-1} \otimes \alpha\mathbf{s}^{-1}\,(\mathcal{D}(\mathbf{t}^{\mathfrak{o}}_k)) + \delta\alpha\mathbf{s}^{-1} \otimes \delta\alpha\mathbf{s}^{-1}\,(\mathcal{D}(\mathbf{t}^{\mathfrak{o}}_k))\big) = 0\,, \tag{7.7}$$

$$\mu\big(\alpha\mathbf{s}^{-1} \otimes \delta\alpha\mathbf{s}^{-1}\,(\mathcal{D}(\mathbf{t}^{\mathfrak{o}}_{m,k})) + \delta\alpha\mathbf{s}^{-1} \otimes \alpha\mathbf{s}^{-1}\,(\mathcal{D}(\mathbf{t}^{\mathfrak{o}}_{m,k})) + \delta\alpha\mathbf{s}^{-1} \otimes \delta\alpha\mathbf{s}^{-1}\,(\mathcal{D}(\mathbf{t}^{\mathfrak{o}}_{m,k}))\big) = 0\,, \tag{7.8}$$

where $\mu$ is the multiplication map:

$$\mu : \mathbb{OP}(\mathsf{KGra}) \to \mathsf{KGra}\,.$$

Boundary conditions for $\alpha$ and $\widetilde{\alpha}$ imply that

$$\delta\alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{c}}_m) = 0\,, \qquad \forall \quad m \ge 2\,, \tag{7.9}$$

$$\delta\alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_k) = 0\,, \qquad \forall \quad k \ge 2\,, \tag{7.10}$$

and

$$\delta\alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{1,k}) = 0\,, \qquad \forall \quad k \geq 0\,. \tag{7.11}$$

Thus there exists $n \geq 2$ such that

$$\delta\alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k}) = 0\,, \qquad \forall \quad m < n,\; k \geq 0\,. \tag{7.12}$$

Vectors

$$\delta\alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k}) \in \mathsf{KGra}(m,k)^{\mathfrak{o}}$$

for $m \geq n$ and $k \geq 0$ will play an important role. For this reason we reserve for them the special notation:

$$\delta\alpha_{m,k} := \delta\alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k})\,. \tag{7.13}$$

Since the restriction

$$\delta\alpha\Big|_{\mathfrak{oc}^{\vee}_{\circ}(m,k)^{\mathfrak{o}}}$$

is $S_m \times S_k$ equivariant, we conclude that

$$\delta\alpha_{m,k} \in \big(\mathsf{KGra}(m,k)^{\mathfrak{o}}\big)^{S_m}\,.$$

Furthermore, using equation (4.6) and the fact that $\delta\alpha$ has degree 1 in the Lie algebra $\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$, we conclude that the vector $\delta\alpha_{m,k}$ has degree zero in

$$\mathbf{s}^{2m-2+k}\big(\mathsf{KGra}(m,k)^{\mathfrak{o}}\big)^{S_m}\,. \tag{7.14}$$

So, from now on, we view $\delta\alpha_{m,k}$ as a vector in (7.14).

Equation (7.12) implies that the components $\mathcal{D}_{\mathsf{Lie}}(\mathbf{t}^{\mathfrak{o}}_{n,k})$, $\mathcal{D}'(\mathbf{t}^{\mathfrak{o}}_{n,k})$ and $\mathcal{D}''(\mathbf{t}^{\mathfrak{o}}_{n,k})$ do not contribute to the left hand side of equation (7.8) for $m = n$.

Hence we have

$$\mu\big(\alpha\mathbf{s}^{-1} \otimes \delta\alpha\mathbf{s}^{-1}\,(\mathcal{D}_{\mathsf{As}}(\mathbf{t}^{\mathfrak{o}}_{n,k})) + \delta\alpha\mathbf{s}^{-1} \otimes \alpha\mathbf{s}^{-1}\,(\mathcal{D}_{\mathsf{As}}(\mathbf{t}^{\mathfrak{o}}_{n,k})) \tag{7.15}$$
$$+\delta\alpha\mathbf{s}^{-1} \otimes \delta\alpha\mathbf{s}^{-1}\,(\mathcal{D}_{\mathsf{As}}(\mathbf{t}^{\mathfrak{o}}_{n,k}))\big) = 0\,.$$

Equation (7.10) implies that

$$\delta\alpha\mathbf{s}^{-1} \otimes \delta\alpha\mathbf{s}^{-1}\;(\mathcal{D}_{\mathsf{As}}(\mathbf{t}^{\mathfrak{o}}_{n,k})) = 0\,.$$

Therefore equation (7.15) is equivalent to

$$\mu\big(\alpha\mathbf{s}^{-1} \otimes \delta\alpha\mathbf{s}^{-1}\,(\mathcal{D}_{\mathsf{As}}(\mathbf{t}^{\mathfrak{o}}_{n,k})) + \delta\alpha\mathbf{s}^{-1} \otimes \alpha\mathbf{s}^{-1}\,(\mathcal{D}_{\mathsf{As}}(\mathbf{t}^{\mathfrak{o}}_{n,k}))\big) = 0\,. \tag{7.16}$$

Using the boundary condition (5.8) and equation (5.10) we rewrite equation (7.16) as follows

$$\Gamma_{\circ\circ} \circ_{2,\mathfrak{o}} \delta\alpha_{n,k-1} - \delta\alpha_{n,k-1} \circ_{1,\mathfrak{o}} \Gamma_{\circ\circ} + \delta\alpha_{n,k-1} \circ_{2,\mathfrak{o}} \Gamma_{\circ\circ} - \dots \tag{7.17}$$
$$+(-1)^{k-1}\delta\alpha_{n,k-1}\circ_{k-1,\mathfrak{o}}\Gamma_{\circ\circ}(-1)^{k}\Gamma_{\circ\circ}\circ_{1,\mathfrak{o}}\delta\alpha_{n,k-1} = 0\,.$$

where

$$\delta\alpha_{n,k-1} = \delta\alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,k-1})\,. \tag{7.18}$$

Thus, we proved the following statement.

**Claim 7.1** *If equation (7.12) holds then, for each $k \ge 0$, the vector $\delta\alpha_{n,k}$ is a degree zero cocycle in the cochain complex*

$$\mathsf{KGra}^{\mathrm{Hoch}}_{\mathrm{inv}} = \mathbf{s}^{2n-2} \bigoplus_{k \ge 0} \mathbf{s}^k \big(\mathsf{KGra}(n,k)^{\mathfrak{o}}\big)^{S_n} \tag{7.19}$$

*with the differential $\partial^{\mathrm{Hoch}}$ given by the formula*

$$\begin{aligned}\partial^{\mathrm{Hoch}}(\gamma) = \Gamma_{\circ\circ} \circ_{2,\mathfrak{o}} \gamma - \gamma \circ_{1,\mathfrak{o}} \Gamma_{\circ\circ} + \gamma \circ_{2,\mathfrak{o}} \Gamma_{\circ\circ} - \dots \\ +(-1)^k \gamma \circ_{k,\mathfrak{o}} \Gamma_{\circ\circ} (-1)^{k+1} \Gamma_{\circ\circ} \circ_{1,\mathfrak{o}} \gamma\,, \\ \gamma \in \mathbf{s}^{2n-2+k} \big(\mathsf{KGra}(n,k)^{\mathfrak{o}}\big)^{S_n}. \qquad \square\end{aligned} \tag{7.20}$$

The cochain complex (7.19) is examined in detail in Appendix A. However, for now, we only need Corollary A.2. Namely, combining Corollary A.2 with Claim 7.1 we easily conclude that

**Claim 7.2** *The white vertex of each graph in the linear combination*

$$\delta\alpha_{n,1} \in \mathbf{s}^{2n-1}\big(\mathsf{KGra}(n,1)^{\mathfrak{o}}\big)^{S_n}$$

*has valency* $1$ . $\square$

### 7.1 Pikes in $\delta\alpha_{n,0}$ can be "killed"

In general linear combinations $\delta\alpha_{m,k}$ (7.13) may contain graphs with a black vertex of valency 1 whose adjacent edge terminates at this vertex. We call such vertices *pikes*.

Provided equation (7.12) holds, we have the following statement.

**Claim 7.3** *In the homotopy class $[\widetilde{F}]$ of a stable formality quasi-isomorphism $\widetilde{F}$ there exists a representative for which all graphs of the linear combination $\delta\alpha_{n,0}$ do not have pikes.*

Proof. Let us denote by $\delta\alpha^r_{n,0}$ the linear combination in $\mathsf{KGra}(n,0)^{\mathfrak{o}}$ which is obtained from $\delta\alpha_{n,0}$ by retaining only graphs with exactly $r$ pikes.

Since $\delta\alpha^r_{n,0}$ is a linear combination of graphs without white vertices, it is a cocycle in the complex (B.1) with the differential $\mathfrak{d}$ (B.5) examined in detail in Appendix B. According to Lemma B.3 from this appendix we have

$$\mathfrak{d}\mathfrak{d}^*(\delta\alpha^r_{n,0}) = r\delta\alpha^r_{n,0}\,. \tag{7.21}$$

Thus, for the vector

$$\chi_{n-1,1} = -\sum_{r\ge 1} \frac{1}{r}\mathfrak{d}^*(\delta\alpha^r_{n,0}) \in \mathbf{s}^{2(n-1)-1}\big(\mathsf{KGra}(n-1,1)^{\mathfrak{o}}\big)^{S_{n-1}}, \tag{7.22}$$

the linear combination

$$\delta\alpha_{n,0} + \mathfrak{d}(\chi_{n-1,1}) \tag{7.23}$$

does not have pikes.

Next, we define the degree 0 vector

$$\xi \in \mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra})$$

by setting

$$\xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n-1,1}) = \chi_{n-1,1}\,, \qquad \xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{c}}_{m_1}) = \xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{k_1}) = \xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k}) = 0 \tag{7.24}$$

for all $m_1$, $k_1$ and for all pairs $(m,k) \neq (n-1,1)$.

Then, we act by $\exp(\mathrm{ad}_\xi)$ on the MC element $\widetilde{\alpha}$ and get a new MC element

$$\widetilde{\alpha}' = \exp(\mathrm{ad}_\xi)\widetilde{\alpha}\,. \tag{7.25}$$

We denote by $\delta\alpha' = \widetilde{\alpha}' - \alpha$ the new difference between MC elements and remark that

$$\delta\alpha' = \exp(\mathrm{ad}_\xi)\delta\alpha + \exp(\mathrm{ad}_\xi)\alpha - \alpha\,. \tag{7.26}$$

To prove the desired claim, we need to verify the following two statements about $\delta\alpha'$:

- First, we should check that
$$\delta\alpha'(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k}) = 0 \tag{7.27}$$
for all $m < n$ and $k \geq 0$;
- Second, we need to show that each graph in the linear combination
$$\delta\alpha'(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,0}) \tag{7.28}$$
has no pikes.

To prove (7.27), we will use the additional descending filtration $\mathcal{F}^{\mathfrak{c}}_\bullet$ by arity of the color $\mathfrak{c}$ (see equation (2.47) in Section 2.3).

Equations (7.9) and (7.12) imply that

$$\delta\alpha \in \mathcal{F}^{\mathfrak{c}}_{n-1}\mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra})\,. \tag{7.29}$$

Furthermore, we have

$$\alpha \in \mathcal{F}^{\mathfrak{c}}_{0}\mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra})\,, \tag{7.30}$$

and

$$\xi \in \mathcal{F}^{\mathfrak{c}}_{n-2}\mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra})\,. \tag{7.31}$$

Therefore equation (7.27) holds obviously for $m < n-1$.

Equations (7.29) and (7.31) imply that

$$\delta\alpha'(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n-1,k}) = \sum_{q=1}^{\infty} \frac{1}{q!}\mathrm{ad}^q_\xi(\alpha)\big(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n-1,k}\big)\,. \tag{7.32}$$

Thus we need to prove that

$$\sum_{q=1}^{\infty} \frac{1}{q!}\mathrm{ad}^q_\xi(\alpha)\big(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n-1,k}\big) = 0\,. \tag{7.33}$$

We leave the easy case $n=2$ to the reader and proceed to the case $n > 2$.

If $n > 2$ then we have the inequality

$$(n-2)(q-1) \geq 1\,, \qquad \forall \;\; q \geq 2$$

which implies that

$$q(n-2) \geq n-1$$

and hence

$$\mathrm{ad}^q_{\xi}(\alpha)\big(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n-1,k}\big) = 0 \qquad \forall \;\; q \geq 2\,.$$

On the other hand,

$$[\xi, \alpha]\big(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n-1,k}\big) = 0$$

for all $k \neq 2$ and

$$[\xi, \alpha]\big(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n-1,2}\big) = \chi_{n-1,1} \circ_{1,\mathfrak{o}} \Gamma_{\circ\circ} - \Gamma_{\circ\circ} \circ_{1,\mathfrak{o}} \chi_{n-1,1} - \Gamma_{\circ\circ} \circ_{2,\mathfrak{o}} \chi_{n-1,1} = 0$$

because the white vertex in each graph of the linear combination $\chi_{n-1,1}$ has valency $1$.

Thus equation (7.27) holds for all $m < n$ and $k \geq 0$.

Let us now prove that each graph in the linear combination (7.28) has no pikes.

It is not hard to see that for every $f \in \mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$

$$[\xi, f](\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,0}) = -(-1)^{|f|} \sum_{i=1}^{n} (\tau_{n,i}, \mathrm{id})\big(\xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n-1,1}) \circ_{1,\mathfrak{o}} f(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{1,0})\big)\,, \tag{7.34}$$

where $\tau_{n,i}$ is the following family of cycles in $S_n$

$$\tau_{n,i} = \begin{pmatrix} 1 & \dots & i-1 & i & \dots & n-1 & n \\ 1 & \dots & i-1 & i+1 & \dots & n & i \end{pmatrix}. \tag{7.35}$$

Therefore $\delta\alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,0})$ and $[\xi, \alpha](\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,0})$ are the only possibly non-zero terms in $\delta\alpha'(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,0})$.

Thus

$$\delta\alpha'(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,0}) = \delta\alpha_{n,0} + \sum_{i=1}^{n} (\tau_{n,i}, \mathrm{id})\big(\xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n-1,1}) \circ_{1,\mathfrak{o}} \alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{1,0})\big) = \tag{7.36}$$

$$\delta\alpha_{n,0} + \sum_{i=1}^{n} (\tau_{n,i}, \mathrm{id})\big(\xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n-1,1}) \circ_{1,\mathfrak{o}} \Gamma^{\mathrm{br}}_{0}\big) =$$

$$\delta\alpha_{n,0} + \mathfrak{d}(\chi_{n-1,1})\,.$$

As we showed above, all graphs in this linear combination do not have pikes.

Claim (7.3) is proved.

□

### 7.2 $\delta\alpha_{n,0}$ is a cocycle in $\mathsf{dfGC}$

Since $\delta\alpha_{n,0} = \delta\alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,0})$ is a linear combination of graphs with only black vertices and it is invariant with respect to the action of $S_n$, we may view $\delta\alpha_{n,0}$ as a vector in $\mathsf{dfGC}$ (6.1). Furthermore, since $\delta\alpha_{n,0}$ has exactly $2n-2$ edges, $\delta\alpha_{n,0}$ has degree zero in $\mathsf{dfGC}$.

Due to Claim 7.3, we may assume without loss of generality that, the linear combination $\delta\alpha_{n,0}$ does not contain graphs with pikes.

We claim that

**Claim 7.4** *Let $\alpha$ and $\widetilde{\alpha}$ be MC elements in (5.6) corresponding to stable formality quasi-isomorphisms. If $\delta\alpha$ satisfies (7.12) and all graphs in the linear combination $\delta\alpha_{n,0}$ do not have pikes, then the vector $\delta\alpha_{n,0}$ is a degree zero cocycle in* $\mathsf{dfGC}$ *(6.1).*

Proof. Let us consider the identity (7.8) for $m = n+1$ and $k = 0$

$$\mu\big(\alpha \mathbf{s}^{-1} \otimes \delta\alpha \mathbf{s}^{-1} (\mathcal{D}(\mathbf{t}^{\mathfrak{o}}_{n+1,0})) + \delta\alpha \mathbf{s}^{-1} \otimes \alpha \mathbf{s}^{-1} (\mathcal{D}(\mathbf{t}^{\mathfrak{o}}_{n+1,0})) \tag{7.37}$$
$$+\delta\alpha \mathbf{s}^{-1} \otimes \delta\alpha \mathbf{s}^{-1} (\mathcal{D}(\mathbf{t}^{\mathfrak{o}}_{n+1,0}))\big) = 0\,.$$

Since

$$\mathcal{D}(\mathbf{t}^{\mathfrak{o}}_{n+1,0}) = \sum_{p=2}^{n+1} \sum_{\tau \in \mathrm{Sh}_{p,n+1-p}} (\tau, \mathrm{id})\big(\mathbf{t}^{\mathfrak{o}}_{n+2-p,0} \circ_{1,\mathfrak{c}} \mathbf{t}^{\mathfrak{c}}_{p}\big) + \tag{7.38}$$
$$-\sum_{r=1}^{n} \sum_{\sigma \in \mathrm{Sh}_{r,n+1-r}} (\sigma, \mathrm{id})\big(\mathbf{t}^{\mathfrak{o}}_{r,1} \circ_{1,\mathfrak{o}} \mathbf{t}^{\mathfrak{o}}_{n+1-r,0}\big)$$

and $\delta\alpha$ satisfies (7.9) and (7.12), we have

$$\delta\alpha \mathbf{s}^{-1} \otimes \delta\alpha \mathbf{s}^{-1} (\mathcal{D}(\mathbf{t}^{\mathfrak{o}}_{n+1,0})) = 0\,,$$

$$\delta\alpha \mathbf{s}^{-1} \otimes \alpha \mathbf{s}^{-1} (\mathcal{D}(\mathbf{t}^{\mathfrak{o}}_{n+1,0})) = \tag{7.39}$$

$$\sum_{\tau \in \mathrm{Sh}_{2,n-1}} (\tau, \mathrm{id})\big(\delta\alpha_{n,0} \circ_{1,\mathfrak{c}} \Gamma_{\bullet\!-\!\bullet}\big) - \sum_{i=1}^{n+1} (\tau_{n+1,i}, \mathrm{id})\big(\delta\alpha_{n,1} \circ_{1,\mathfrak{o}} \Gamma^{\mathrm{br}}_{0}\big)\,,$$

and

$$\alpha \mathbf{s}^{-1} \otimes \delta\alpha \mathbf{s}^{-1} (\mathcal{D}(\mathbf{t}^{\mathfrak{o}}_{n+1,0})) = -\sum_{i=1}^{n+1} (\sigma_{n+1,i}, \mathrm{id})\big(\Gamma^{\mathrm{br}}_{1} \circ_{1,\mathfrak{o}} \delta\alpha_{n,0}\big)\,, \tag{7.40}$$

where $\sigma_{n+1,i}$ is the family of cycles

$$\sigma_{n+1,i} = \begin{pmatrix} 1 & 2 & \dots & i & i+1 & \dots & n+1 \\ i & 1 & \dots & i-1 & i+1 & \dots & n+1 \end{pmatrix}\,, \tag{7.41}$$

in $S_{n+1}$ and $\tau_{n+1,i}$ is defined in (7.35).

Thus we get an identity which involves both $\delta\alpha_{n,0}$ and $\delta\alpha_{n,1}$:

$$\sum_{\tau \in \mathrm{Sh}_{2,n-1}} (\tau, \mathrm{id})\big(\delta\alpha_{n,0} \circ_{1,\mathfrak{c}} \Gamma_{\bullet\!-\!\bullet}\big) - \sum_{i=1}^{n+1} (\sigma_{n+1,i}, \mathrm{id})\big(\Gamma^{\mathrm{br}}_{1} \circ_{1,\mathfrak{o}} \delta\alpha_{n,0}\big) \tag{7.42}$$
$$-\sum_{i=1}^{n+1} (\tau_{n+1,i}, \mathrm{id})\big(\delta\alpha_{n,1} \circ_{1,\mathfrak{o}} \Gamma^{\mathrm{br}}_{0}\big) = 0\,.$$

Notice that $\Gamma^{\mathrm{br}}_0$ consists of a single black vertex and the insertion $\delta\alpha_{n,1} \circ_{1,\mathfrak{o}} \Gamma^{\mathrm{br}}_0$ is nothing but replacing the single white vertex in each graph of the linear combination $\delta\alpha_{n,1}$ by black vertex with label $n+1$.

On the other hand, Claim 7.2 says that all white vertices in $\delta\alpha_{n,1}$ have valency $1$. Thus, for each graph in $\delta\alpha_{n,1} \circ_{1,\mathfrak{o}} \Gamma^{\mathrm{br}}_0$ the black vertex with label $n+1$ is necessarily a pike.

Since $\delta\alpha_{n,0}$ does not have pikes, the sum

$$-\sum_{i=1}^{n+1} (\tau_{n+1,i}, \mathrm{id})\big(\delta\alpha_{n,1} \circ_{1,\mathfrak{o}} \Gamma^{\mathrm{br}}_{0}\big)\,.$$

should necessarily cancel the all the graphs with pikes in the sum

$$\sum_{\tau\in \mathrm{Sh}_{2,n-1}} (\tau, \mathrm{id})\big(\delta\alpha_{n,0}\circ_{1,\mathfrak{c}} \Gamma_{\bullet\!-\!\bullet}\big)\,. \tag{7.43}$$

It is not hard to see that the graphs with pikes in the expression (7.43) form the following linear combination[8]

$$\sum_{i=1}^{n+1} (\tau_{n+1,i}, \mathrm{id})\big(\Gamma^{>}_{\bullet\!-\!\bullet}\circ_{1,\mathfrak{c}} \delta\alpha_{n,0}\big)\,, \tag{7.44}$$

where $\tau_{n+1,i}$ is defined in (7.35) and

$$\Gamma^{>}_{\bullet\!-\!\bullet} = \overset{1}{\bullet}\longrightarrow\overset{2}{\bullet}\,. \tag{7.45}$$

Thus we conclude that

$$-\sum_{i=1}^{n+1} (\tau_{n+1,i}, \mathrm{id})\big(\delta\alpha_{n,1}\circ_{1,\mathfrak{o}} \Gamma^{\mathrm{br}}_{0}\big) = -\sum_{i=1}^{n+1} (\tau_{n+1,i}, \mathrm{id})\big(\Gamma^{>}_{\bullet\!-\!\bullet}\circ_{1,\mathfrak{c}} \delta\alpha_{n,0}\big)\,, \tag{7.46}$$

On the other hand, we have

$$-\sum_{i=1}^{n+1} (\sigma_{n+1,i}, \mathrm{id})\big(\Gamma^{\mathrm{br}}_{1}\circ_{1,\mathfrak{o}} \delta\alpha_{n,0}\big) = -\sum_{i=1}^{n+1} (\tau_{n+1,i}, \mathrm{id})\big(\Gamma^{<}_{\bullet\!-\!\bullet}\circ_{1,\mathfrak{c}} \delta\alpha_{n,0}\big)\,, \tag{7.47}$$

where

$$\Gamma^{<}_{\bullet\!-\!\bullet} = \overset{1}{\bullet}\longleftarrow\overset{2}{\bullet}\,. \tag{7.48}$$

Therefore identity (7.42) implies that

$$\sum_{\tau\in \mathrm{Sh}_{2,n-1}} (\tau, \mathrm{id})\big(\delta\alpha_{n,0}\circ_{1,\mathfrak{c}} \Gamma_{\bullet\!-\!\bullet}\big) - \sum_{i=1}^{n+1} (\tau_{n+1,i}, \mathrm{id})\big(\Gamma^{<}_{\bullet\!-\!\bullet}\circ_{1,\mathfrak{c}} \delta\alpha_{n,0}\big) - \sum_{i=1}^{n+1} (\tau_{n+1}, \mathrm{id})\big(\Gamma^{>}_{\bullet\!-\!\bullet}\circ_{1,\mathfrak{c}} \delta\alpha_{n,0}\big) = 0\,.$$

In other words, $\delta\alpha_{n,0}$ is indeed a cocycle in $\mathsf{dfGC}$ (6.1).

□

Claim 7.4 has the following corollary.

**Corollary 7.1** *Let $\alpha$ and $\widetilde{\alpha}$ be MC elements in* (5.6) *corresponding to stable formality quasi-isomorphisms. If $\delta\alpha$ satisfies* (7.12)*, and all graphs in the linear combination $\delta\alpha_{n,0}$ do not have pikes, then there exists degree zero cocycle $\gamma\in\mathsf{dfGC}$ such that*

$$\Big(\widetilde{\alpha} \quad - \quad \big(\exp(\mathrm{ad}_{J(\gamma)})\alpha\big)\Big)\,(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,0}) = 0\,, \tag{7.49}$$

*and*

$$\Big(\widetilde{\alpha} \quad - \quad \big(\exp(\mathrm{ad}_{J(\gamma)})\alpha\big)\Big)\,(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k}) = 0 \tag{7.50}$$

*for all $m<n$ and $k\geq 0$.*

[8] Here we use the fact that $\delta\alpha_{n,0}$ carries an even degree.

Proof. Due to Claim 7.4 the linear combination $\delta\alpha_{n,0}$ is a degree zero cocycle in $\mathsf{dfGC}$. So we set

$$\gamma = \delta\alpha_{n,0}\,. \tag{7.51}$$

In other words,

$$\gamma(1_m) = \begin{cases} -\delta\alpha_{n,0} & \text{if} \quad m = n \quad \text{and} \\ 0 & \text{otherwise}\,, \end{cases}$$

where $1_m$ denotes the generator

$$\mathbf{s}^{2-2m}1 \in \mathbf{s}^{2-2m}\mathbb{K} \cong \Lambda^2\mathsf{coCom}(m)\,.$$

Thus, for any degree 1 element $f \in \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$ we have

$$[J(\gamma), f](\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,0}) = -\mu\big(f\mathbf{s}^{-1} \otimes J(\gamma)\mathbf{s}^{-1}\,(\mathbf{t}^{\mathfrak{o}}_{1,0} \circ_{1,\mathfrak{c}} \mathbf{t}^{\mathfrak{c}}_{n})\big) = \tag{7.52}$$

$$-f(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{1,0}) \circ_{1,\mathfrak{c}} \gamma$$

and

$$[J(\gamma), f](\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{1,0}) = 0\,. \tag{7.53}$$

Therefore,

$$\big(\exp(\mathrm{ad}_{J(\gamma)})\alpha\big)\,(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,0}) = \alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,0}) - \alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{1,0}) \circ_{1,\mathfrak{c}} \gamma = \tag{7.54}$$

$$+\delta\alpha_{n,0} - \Gamma^{\mathrm{br}}_0 \circ_{1,\mathfrak{c}} \gamma = \delta\alpha_{n,0} - \gamma = 0$$

and equation (7.49) follows.

In addition, since

$$J(\gamma) \in \mathcal{F}^{\mathfrak{c}}_{n-1}\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}),$$

we conclude that (7.50) holds as well.

□

### 7.3 If $\delta\alpha_{n,0} = 0$ then the remaining vectors $\delta\alpha_{n,k}$ can be "killed" by adjusting the representative $\widetilde{F}$

Due to Corollary 7.1, we can assume, without the loss of generality, that $\delta\alpha_{n,0} = 0$. In this section we prove the following statement.

**Claim 7.5** *If $\delta\alpha$ satisfies* (7.12), *and*

$$\delta\alpha_{n,0} = 0 \tag{7.55}$$

*then there exists a degree zero vector $\xi \in \mathcal{F}_{n-1}\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$ satisfying condition* (5.13) *and such that*

$$\big(\exp(\mathrm{ad}_\xi)(\widetilde{\alpha}) - \alpha\big)\,(\mathbf{t}^{\mathfrak{o}}_{m,k}) = 0 \qquad \forall \qquad m \le n, \quad k \ge 0\,. \tag{7.56}$$

Proof. Suppose that $\delta\alpha_{n,k} \neq 0$ for some $k$. Hence there exists $k' \geq 1$ such that

$$\delta\alpha_{n,k} = 0 \qquad \forall \quad k < k' \tag{7.57}$$

or equivalently

$$\widetilde{\alpha}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,k}) = \alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,k}) \qquad \forall \quad k < k'\,. \tag{7.58}$$

We will now prove that $\delta\alpha_{n,k'}$ can be "killed" by switching to a MC element isomorphic to $\widetilde{\alpha}$ via

$$\exp(\xi_{k'})$$

where $\xi_{k'}$ is a degree zero vector in $\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$ satisfying (5.13) and

$$\xi_{k'} \in \mathcal{F}_{n+k'-2}\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})\,.$$

The proof of this fact consists of two steps. In the first step, we switch to a MC element $\widetilde{\alpha}^{(1)}$ which is isomorphic to $\widetilde{\alpha}$ and such that

$$\widetilde{\alpha}^{(1)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k}) - \alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k}) = 0 \qquad \forall \;\; m < n,\;\; k \geq 0\,, \tag{7.59}$$

$$\widetilde{\alpha}^{(1)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,k}) - \alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,k}) = 0 \qquad k < k'\,, \tag{7.60}$$

and the vector

$$\widetilde{\alpha}^{(1)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,k'}) - \alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,k'}) \tag{7.61}$$

satisfies Properties A.1, A.2, i.e. for each graph in (7.61) white vertices have valency 1 and the linear combination (7.61) is anti-symmetric with respect to permutations of labels on white vertices.

In the second step we switch to a MC element $\widetilde{\alpha}^{(2)}$ which is isomorphic to $\widetilde{\alpha}^{(1)}$ and such that

$$\widetilde{\alpha}^{(2)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k}) - \alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k}) = 0 \qquad \forall \;\; m < n,\;\; k \geq 0\,, \tag{7.62}$$

$$\widetilde{\alpha}^{(2)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,k}) - \alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,k}) = 0 \qquad k \leq k'\,. \tag{7.63}$$

The second step is based upon an auxiliary construction described in Appendix B.

**Step 1.** If $k' = 1$ then the linear combination $\delta\alpha_{n,k'}$ already satisfies Properties A.1, A.2 due to Claim 7.2. So, in this case, we proceed to Step 2. In Step 1, it remains to consider the case $k' \geq 2$.

Due to Claim 7.1, the vector

$$\delta\alpha_{n,k'} \in \mathbf{s}^{2n-2+k'}\Big(\mathsf{KGra}(n,k')^{\mathfrak{o}}\Big)^{S_n} \tag{7.64}$$

is a cocycle in the complex (A.1) with the differential (A.2). Thus, Corollary A.1 implies that there exists a vector

$$\psi_{n,k'-1} \in \mathbf{s}^{2n+k'-3}\Big(\mathsf{KGra}(n,k'-1)^{\mathfrak{o}}\Big)^{S_n} \tag{7.65}$$

such that

$$\delta\alpha_{n,k'} - \partial^{\mathrm{Hoch}}\psi_{n,k'-1} \tag{7.66}$$

satisfies Properties A.1, A.2, i.e. for each graph in (7.66) white vertices have valency 1 and the linear combination (7.66) is anti-symmetric with respect to permutations of labels on white vertices.

So we form a degree zero vector

$$\xi^1_{k'} \in \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$$

by setting

$$\xi^1_{k'}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,k'-1}) = \psi_{n,k'-1}\,, \qquad \xi^1_{k'}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{c}}_{m_1}) = \xi^1_{k'}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{k_1}) = \xi^1_{k'}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k}) = 0 \tag{7.67}$$

for all $m_1 \ge 2$, $k_1 \ge 2$, and all pairs $(m,k) \neq (n, k'-1)$.

It is obvious that

$$\xi^1_{k'} \in \mathcal{F}_{n+k'-2}\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})\,, \tag{7.68}$$

and

$$\xi^1_{k'} \in \mathcal{F}^{\mathfrak{c}}_{n-1}\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})\,. \tag{7.69}$$

Next we consider the MC element

$$\widetilde{\alpha}^{(1)} = \exp(\mathrm{ad}_{\xi^1_{k'}})(\widetilde{\alpha}) \tag{7.70}$$

isomorphic to $\widetilde{\alpha}$ via $\exp(\mathrm{ad}_{\xi^1_{k'}})$ and denote by $\delta\alpha^{(1)}$ the difference:

$$\delta\alpha^{(1)} = \widetilde{\alpha}^{(1)} - \alpha\,. \tag{7.71}$$

Expression (7.71) can be rewritten as

$$\delta\alpha^{(1)} = \exp(\mathrm{ad}_{\xi^1_{k'}})(\delta\alpha) + \exp(\mathrm{ad}_{\xi^1_{k'}})(\alpha) - \alpha\,. \tag{7.72}$$

Using (7.72) and inclusions (7.68), (7.69) we conclude that

$$\delta\alpha^{(1)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k}) = 0 \qquad \forall \;\; m < n, \quad k \ge 0\,, \tag{7.73}$$

and

$$\delta\alpha^{(1)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,k}) = 0 \qquad \forall \;\; k \le k'-2\,.$$

Furthermore, using the inequality $n + k' - 2 \ge 2$, it is not hard see that

$$\delta\alpha^{(1)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,k'-1}) = [\xi^1_{k'}, \alpha](\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,k'-1})$$

and a direct computation shows that $[\xi^1_{k'}, \alpha](\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,k'-1}) = 0$. Thus

$$\delta\alpha^{(1)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,k}) = 0\,, \qquad \forall \;\; k < k'\,. \tag{7.74}$$

Finally, for $\mathbf{t}^{\mathfrak{o}}_{n,k'}$ we get

$$\delta\alpha^{(1)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,k'}) = \delta\alpha_{n,k'} + [\xi^1_{k'}, \alpha](\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,k'}) =$$

$$\delta\alpha_{n,k'} + \sum_{p=0}^{k'-2} (-1)^p \xi^1_{k'}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,k'-1}) \circ_{p+1,\mathfrak{o}} \alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_2)$$

$$-\alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_2) \circ_{2,\mathfrak{o}} \xi^1_{k'}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,k'-1}) + (-1)^{k'-1}\alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_2) \circ_{1,\mathfrak{o}} \xi^1_{k'}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,k'-1}) =$$

$$\delta\alpha_{n,k'} - \partial^{\rm Hoch}\psi_{n,k'-1}\,.$$

Thus $\delta\alpha^{(1)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,k'})$ satisfies Properties A.1, A.2.

We can now proceed to Step 2.

**Step 2.** Let us introduce the notation

$$\delta\alpha^{(1)}_{m,k} := \delta\alpha^{(1)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k})$$

and consider identity (7.8) for $m = n+1$, $k = k'-1$ and $\delta\alpha$ replaced by $\delta\alpha^{(1)}$.

Using (7.73), (7.74) together with the identities

$$\delta\alpha^{(1)}(\mathbf{t}^{\mathfrak{c}}_m) = \delta\alpha^{(1)}(\mathbf{t}^{\mathfrak{o}}_k) = 0 \qquad \forall\ \ m\ge 2,\ k\ge 2$$

we deduce that

$$\mu\big(\delta\alpha^{(1)}\mathbf{s}^{-1}\otimes\delta\alpha^{(1)}\mathbf{s}^{-1}\ \mathcal{D}(\mathbf{t}^{\mathfrak{o}}_{n+1,k'-1})\big) = 0\,, \tag{7.75}$$

$$\mu\big(\delta\alpha^{(1)}\mathbf{s}^{-1}\otimes\alpha\mathbf{s}^{-1}\ \mathcal{D}(\mathbf{t}^{\mathfrak{o}}_{n+1,k'-1})\big) = \tag{7.76}$$

$$-\quad\sum_{p=0}^{k'-3}(-1)^p\delta\alpha^{(1)}_{n+1,k'-2}\circ_{p+1,\mathfrak{o}}\Gamma_{\circ\circ} - \sum_{i=1}^{n+1}\sum_{p=0}^{k'-1}(-1)^p\,(\tau_{n+1,i},{\rm id})\big(\delta\alpha^{(1)}_{n,k'}\circ_{p+1,\mathfrak{o}}\Gamma^{\rm br}_0\big)\,,$$

and

$$\mu\big(\alpha\mathbf{s}^{-1}\otimes\delta\alpha^{(1)}\mathbf{s}^{-1}\ \mathcal{D}(\mathbf{t}^{\mathfrak{o}}_{n+1,k'-1})\big) = \Gamma_{\circ\circ}\circ_{2,\mathfrak{o}}\delta\alpha^{(1)}_{n+1,k'-2} \tag{7.77}$$
$$+(-1)^{k'-1}\Gamma_{\circ\circ}\circ_{1,\mathfrak{o}}\delta\alpha^{(1)}_{n+1,k'-2}\,,$$

where $\tau_{n+1,i}$ is defined in (7.35).

Combining all these terms we get the following identity:

$$\sum_{i=1}^{n+1}\sum_{p=0}^{k'-1}(-1)^p\,(\tau_{n+1,i},{\rm id})\big(\delta\alpha^{(1)}_{n,k'}\circ_{p+1,\mathfrak{o}}\Gamma^{\rm br}_0\big) = \Gamma_{\circ\circ}\circ_{2,\mathfrak{o}}\delta\alpha^{(1)}_{n+1,k'-2} + \tag{7.78}$$
$$\sum_{p=0}^{k'-3}(-1)^{p+1}\delta\alpha^{(1)}_{n+1,k'-2}\circ_{p+1,\mathfrak{o}}\Gamma_{\circ\circ} + (-1)^{k'-1}\Gamma_{\circ\circ}\circ_{1,\mathfrak{o}}\delta\alpha^{(1)}_{n+1,k'-2}\,.$$

Thus the vector

$$\rho_{n+1,k'-1} = \sum_{i=1}^{n+1}\sum_{p=1}^{k'}(-1)^p\,(\tau_{n+1,i},{\rm id})\big(\delta\alpha^{(1)}_{n,k'}\circ_{p,\mathfrak{o}}\Gamma^{\rm br}_0\big) \tag{7.79}$$

is $\partial^{\rm Hoch}$-exact in

$$\mathbf{s}^{2(n+1)-2+(k'-1)}\big(\mathsf{KGra}(n+1,k'-1)\big)^{S_{n+1}}\,. \tag{7.80}$$

Using the antisymmetry of $\delta\alpha^{(1)}_{n,k'}$ with respect to the action of $S_{k'}$ on the labels of white vertices, we see that

$$\sum_{p=1}^{k'}(-1)^p\,(\tau_{n+1,i},{\rm id})\big(\delta\alpha^{(1)}_{n,k'}\circ_{p,\mathfrak{o}}\Gamma^{\rm br}_0\big) = k'\,(\tau_{n+1,i},{\rm id})\big(\delta\alpha^{(1)}_{n,k'}\circ_{1,\mathfrak{o}}\Gamma^{\rm br}_0\big)\,.$$

Therefore the expression (7.79) can be rewritten as

$$\rho_{n+1,k'-1} = \mathfrak{d}(\delta\alpha^{(1)}_{n,k'})\,, \tag{7.81}$$

where $\mathfrak{d}$ is the operation defined in (B.5) in Appendix B.

Hence $\rho_{n+1,k'-1}$ is a vector in (7.80) satisfying Properties A.1, A.2. Combining this observation with the fact that $\rho_{n+1,k'-1}$ is $\partial^{\mathrm{Hoch}}$-exact and using the second claim in Corollary A.1 we conclude that

$$\rho_{n+1,k'-1} = 0\,.$$

In other words, $\delta\alpha^{(1)}_{n,k'}$ is a cocycle in the cochain complex (B.1) with the differential $\mathfrak{d}$ (B.5).

Since $k' \ge 1$, Corollary B.1 from Appendix B implies that there exists a vector (of degree $-1$)

$$\psi_{n-1,k'+1} \in \mathbf{s}^{2(n-1)-2+k'+1}\big(\mathsf{KGra}(n-1,k'+1)^{\mathfrak{o}}\big)^{S_{n-1}} \tag{7.82}$$

which satisfies Properties A.1, A.2 and such that

$$\delta\alpha^{(1)}_{n,k'} = (k'+1)\sum_{i=1}^{n}(\tau_{n,i},\mathrm{id})\big(\psi_{n-1,k'+1}\circ_{1,\mathfrak{o}}\Gamma^{\mathrm{br}}_{0}\big)\,. \tag{7.83}$$

Using $\psi_{n-1,k'+1}$ we define the following degree zero vector

$$\xi^2_{k'} \in \mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ},\mathsf{KGra})$$

by setting

$$\xi^2_{k'}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n-1,k'+1}) = -\psi_{n-1,k'+1}\,, \qquad \xi^2_{k'}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{c}}_{m_1}) = \xi^2_{k'}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{k_1}) = \xi^2_{k'}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k}) = 0 \tag{7.84}$$

for all $m_1$, $k_1$ and for all pairs $(m,k) \neq (n-1,k'+1)$.

It is obvious that

$$\xi^2_{k'} \in \mathcal{F}_{n+k'-1}\mathrm{Conv}(\mathfrak{oc}^{\vee},\mathsf{KGra})\,, \tag{7.85}$$

and

$$\xi^2_{k'} \in \mathcal{F}^{\mathfrak{c}}_{n-2}\mathrm{Conv}(\mathfrak{oc}^{\vee},\mathsf{KGra})\,. \tag{7.86}$$

Next we consider the MC element

$$\widetilde{\alpha}^{(2)} = \exp(\mathrm{ad}_{\xi^2_{k'}})(\widetilde{\alpha}^{(1)}) \tag{7.87}$$

isomorphic to $\widetilde{\alpha}^{(1)}$ via $\exp(\mathrm{ad}_{\xi^2_{k'}})$ and rewrite the difference

$$\delta\alpha^{(2)} = \widetilde{\alpha}^{(2)} - \alpha \tag{7.88}$$

as follows

$$\delta\alpha^{(2)} = \exp(\mathrm{ad}_{\xi^2_{k'}})(\delta\alpha^{(1)}) + \exp(\mathrm{ad}_{\xi^2_{k'}})(\alpha) - \alpha\,. \tag{7.89}$$

Inclusion (7.86) implies that

$$\delta\alpha^{(2)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k}) = 0 \qquad \forall\;\; m < n-1,\;\; k \ge 0\,.$$

Furthermore, using the inclusion

$$\delta\alpha^{(1)} \in \mathcal{F}_{n-1}\mathrm{Conv}(\mathfrak{oc}^{\vee},\mathsf{KGra})$$

we conclude that

$$\delta\alpha^{(2)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n-1,k}) = \sum_{p=1}^{\infty} \frac{1}{p!}\mathrm{ad}^{p}_{\xi^2_{k'}}(\alpha)\big(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n-1,k}\big)$$

Let us, first, consider the case $n = 2$. In this case

$$\mathrm{ad}^{p}_{\xi^2_{k'}}(\alpha)\big(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{1,k}\big) = 0 \qquad \forall \;\; p \ge 2 \tag{7.90}$$

simply because

$$\xi^2_{k'}(\mathbf{t}^{\mathfrak{o}}_{q}) = [\xi^2_{k'}, f](\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{q}) = 0$$

for all $q \ge 2$ and for all

$$f \in \mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra})\,.$$

Thus

$$\delta\alpha^{(2)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{1,k}) = [\xi^2_{k'}, \alpha]\big(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{1,k}\big)\,. \tag{7.91}$$

If $n > 2$ then we have the inequality

$$(n-2)(p-1) \ge 1$$

which implies that

$$p(n-2) \ge n-1\,.$$

Hence inclusion (7.86) implies that

$$\mathrm{ad}^{p}_{\xi^2_{k'}}(\alpha)\big(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n-1,k}\big) = 0 \qquad \forall \;\; p \ge 2$$

and we have

$$\delta\alpha^{(2)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n-1,k}) = [\xi^2_{k'}, \alpha]\big(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n-1,k}\big)\,. \tag{7.92}$$

Combining (7.91) with (7.92) we conclude that equation (7.92) holds for all $n \ge 2$.

For the right hand side of (7.92) we have

$$[\xi^2_{k'}, \alpha]\big(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n-1,k}\big) =$$

$$\sum_{p=0}^{k'-2}(-1)^p \xi^2_{k'}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n-1,k-1}) \circ_{p+1,\mathfrak{o}} \alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{2})$$

$$-\alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{2}) \circ_{2,\mathfrak{o}} \xi^2_{k'}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n-1,k-1}) + (-1)^{k-1}\alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{2}) \circ_{1,\mathfrak{o}} \xi^2_{k'}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n-1,k-1})$$

$$= \partial^{\mathrm{Hoch}}\xi^2_{k'}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n-1,k-1})\,.$$

The expression $\partial^{\mathrm{Hoch}}\xi^2_{k'}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n-1,k-1}) = 0$ for $k \neq k'+2$ because $\xi^2_{k'}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n-1,q}) = 0$ whenever $q \neq k'+1$ and

$$\partial^{\mathrm{Hoch}}\xi^2_{k'}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n-1,k'+2}) = 0$$

because the vector $\psi_{n-1,k'+1}$ satisfies Properties A.1, A.2 and hence is $\partial^{\mathrm{Hoch}}$-closed.

Thus we conclude that

$$\delta\alpha^{(2)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k}) = 0 \qquad \forall \;\; m \le n-1, \;\; k \ge 0\,. \tag{7.93}$$

To complete Step 2 it remains to show that

$$\delta\alpha^{(2)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,k}) = 0 \qquad \forall \;\; k \le k'\,. \tag{7.94}$$

For $k < k'$ equation (7.94) follows immediately from the inclusion (7.85) and for $k = k'$ we get

$$\delta\alpha^{(2)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,k'}) = \delta\alpha^{(1)}_{n,k'} + [\xi^2_{k'}, \alpha]\big(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,k'}\big) =$$

$$\delta\alpha^{(1)}_{n,k'} + \sum_{p=0}^{k}\sum_{i=1}^{n}(-1)^p(\tau_{n,i}, \mathrm{id})\Big(\xi^2_{k'}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n-1,k'+1}) \circ_{p+1,\mathfrak{o}} \alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{1,0})\Big) =$$

$$\delta\alpha^{(1)}_{n,k'} - \sum_{p=0}^{k'}\sum_{i=1}^{n}(-1)^p(\tau_{n,i}, \mathrm{id})\big(\psi_{n-1,k'+1} \circ_{p+1,\mathfrak{o}} \Gamma^{\mathrm{br}}_0\big)\,.$$

On the other hand, since $\psi_{n-1,k'+1}$ is anti-symmetric with respect to permutations of labels on white vertices,

$$\sum_{p=0}^{k'}\sum_{i=1}^{n}(-1)^p(\tau_{n,i}, \mathrm{id})\big(\psi_{n-1,k'+1} \circ_{p+1,\mathfrak{o}} \Gamma^{\mathrm{br}}_0\big) = (k'+1)\sum_{i=1}^{n}(\tau_{n,i}, \mathrm{id})\big(\psi_{n-1,k'+1} \circ_{1,\mathfrak{o}} \Gamma^{\mathrm{br}}_0\big)$$

Hence,

$$\delta\alpha^{(2)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,k'}) = 0$$

due to equation (7.83) and Step 2 is complete.

Summarizing the results of Step 1 and Step 2, we conclude that there exists vectors

$$\xi^1_{k'} \in \mathcal{F}_{n+k'-2}\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})\,, \qquad \xi^2_{k'} \in \mathcal{F}_{n+k'-1}\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})\,,$$

such that

$$\big(\exp(\mathrm{ad}_{\xi^2_{k'}})\exp(\mathrm{ad}_{\xi^1_{k'}})(\widetilde{\alpha}) - \alpha\big)\,(\mathbf{t}^{\mathfrak{o}}_{m,k}) = 0 \qquad \forall \qquad m < n, \quad k \ge 0 \tag{7.95}$$

and

$$\big(\exp(\mathrm{ad}_{\xi^2_{k'}})\exp(\mathrm{ad}_{\xi^1_{k'}})(\widetilde{\alpha}) - \alpha\big)\,(\mathbf{t}^{\mathfrak{o}}_{n,k}) = 0 \qquad \forall \quad k \le k'\,. \tag{7.96}$$

Thus, for

$$\xi_{k'} = \mathrm{CH}(\xi^2_{k'}, \xi^1_{k'}) \in \mathcal{F}_{n+k'-2}\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}) \tag{7.97}$$

we get

$$\big(\exp(\mathrm{ad}_{\xi_{k'}})(\widetilde{\alpha}) - \alpha\big)\,(\mathbf{t}^{\mathfrak{o}}_{m,k}) = 0 \qquad \forall \qquad m < n, \quad k \ge 0\,, \tag{7.98}$$

and

$$\big(\exp(\mathrm{ad}_{\xi_{k'}})(\widetilde{\alpha}) - \alpha\big)\,(\mathbf{t}^{\mathfrak{o}}_{n,k}) = 0 \qquad \forall \quad k \le k'\,. \tag{7.99}$$

Claim 7.5 can now be proved by induction on $k'$. Indeed, by setting[9]

$$\xi = \lim_{k'\to\infty} \mathrm{CH}(\xi_{k'}, \mathrm{CH}(\xi_{k'-1}, (\dots, \mathrm{CH}(\xi_2, \xi_1))\dots) \tag{7.100}$$

we get an degree 0 element

$$\xi \in \mathcal{F}_{n-1}\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$$

such that (7.56) holds.

Claim 7.5 is proved.

$\square$

Here is the upshot of the above intermediate steps:

[9]The limit (7.100) exists because $\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$ is complete with respect to the filtration by arity.

**Claim 7.6** *If $\alpha$ and $\widetilde{\alpha}$ are MC elements of* $\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$ *corresponding to stable formality quasi-isomorphisms and there is $n \ge 2$ such that*

$$(\widetilde{\alpha} - \alpha)\big(\mathbf{s}^{-1}\, \mathbf{t}^{\mathfrak{o}}_{m,k}\big) = 0 \qquad \forall \;\; m < n, \quad k \ge 0\,, \tag{7.101}$$

*then there exists a degree* 0 *vector*

$$\xi^{(n)} \in \mathcal{F}_{n-1}\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}) \tag{7.102}$$

*satisfying* (5.13) *and* (5.14) *and a degree zero cocycle*

$$\gamma^{(n)} \in \mathcal{F}_{n-1}\mathsf{dfGC} \tag{7.103}$$

*such that*

$$\big(\exp(\mathrm{ad}_{\xi^{(n)}})\widetilde{\alpha} - \exp(\mathrm{ad}_{J(\gamma^{(n)})})\alpha\big)\,\big(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k}\big) = 0 \qquad \forall \;\; m < n+1, \quad k \ge 0\,. \tag{7.104}$$

$\square$

Iterating this argument infinitely many times, we conclude that the MC element $\alpha$ is connected to the MC element

$$\lim_{N\to\infty} \exp(\mathrm{ad}_{\xi^{(N)}}) \dots \exp(\mathrm{ad}_{\xi^{(n+1)}}) \exp(\mathrm{ad}_{\xi^{(n)}})\widetilde{\alpha} \tag{7.105}$$

via the action of element

$$\lim_{N\to\infty} \exp(\mathrm{ad}_{J(\gamma^{(N)})}) \dots \exp(\mathrm{ad}_{J(\gamma^{(n+1)})}) \exp(\mathrm{ad}_{J(\gamma^{(n)})})\,. \tag{7.106}$$

The limits in (7.105) and (7.106) exist due to completeness of $\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$ and $\mathsf{dfGC}$ with respect to the filtration by "arity".

We proved that the action of $\exp\big(H^0(\mathsf{dfGC})\big)$ on homotopy classes of stable formality quasi-isomorphisms is transitive.

# 8 The action of $\exp\big(H^0(\mathsf{dfGC})\big)$ is faithful

Let $\alpha$ be a MC element of $\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$ representing a stable formality quasi-isomorphism. Furthermore, let $\gamma$ be a degree zero cocycle in $\mathsf{dfGC}$. Let us assume that there exists a degree zero vector

$$\xi \in \mathcal{F}_1\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}) \tag{8.1}$$

which satisfies conditions (5.13), (5.14) and such that

$$\exp\big(\mathrm{ad}_{J(\gamma)}\big)\alpha = \exp\big(\mathrm{ad}_{\xi}\big)\alpha\,. \tag{8.2}$$

Our goal is to show that $\gamma$ is exact.

Let us assume that $\gamma$ is not exact and try to arrive at a contradiction.

If $\gamma$ is not exact, then there exists an integer $n \ge 3$ (every non-trivial degree zero cocycle is cocycle lies in $\mathcal{F}_2\mathsf{dfGC}$) such that $\gamma$ is cohomologous to a cocycle $\gamma' \in \mathcal{F}_{n-1}\mathsf{dfGC}$ but is not cohomologous to any cocycle in $\mathcal{F}_n\mathsf{dfGC}$.

Thus we may assume, without loss of generality, that[10]

$$\gamma \in \mathcal{F}_{n-1}\mathsf{dfGC} \tag{8.3}$$

[10] i.e. all graphs in $\gamma$ have $\ge n$ vertices.

and the linear combination of all graphs with exactly $n$ vertices in $\gamma$ forms a non-trivial cocycle in $\mathsf{dfGC}$.

Furthermore, due to Remark 6.1, we may also assume, without loss of generality, that

**Condition 8.1** *Every graph in the linear combination $\gamma$ does not have pikes.*

Equations (5.14), and (5.17) imply that

$$\xi \in \mathcal{F}^{\mathfrak{c}}_1 \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})\,. \tag{8.4}$$

Hence, there exists an integer $m \ge 2$ such that

$$\xi \in \mathcal{F}^{\mathfrak{c}}_{m-1} \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})\,. \tag{8.5}$$

In other words,

$$\xi(\mathbf{s}^{-1}\, \mathbf{t}^{\mathfrak{o}}_{m',k}) = 0 \qquad \forall \;\; m' \le m-1, \quad k \ge 0\,. \tag{8.6}$$

Since the filtration $\mathcal{F}^{\mathfrak{c}}_{\bullet}$ is decreasing, we may assume, without loss of generality, that

$$m \le n-1\,. \tag{8.7}$$

So now we have equation (8.2), where $\gamma$ is a degree zero cocycle in $\mathcal{F}_{n-1}\mathsf{dfGC}$, $\xi$ is a degree zero vector in $\mathcal{F}^{\mathfrak{c}}_{m-1} \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$ satisfying conditions (5.13), (5.14) and

$$2 \le m \le n-1\,.$$

The vectors $\xi(\mathbf{s}^{-1}\, \mathbf{t}^{\mathfrak{o}}_{m,k})$ will play a special role. So we reserve for them the special notation

$$\xi_{m,k} := \xi(\mathbf{s}^{-1}\, \mathbf{t}^{\mathfrak{o}}_{m,k})\,. \tag{8.8}$$

## 8.1 The vectors $\xi_{m,k}$ are $\partial^{\mathrm{Hoch}}$-closed

We claim that

**Claim 8.1** *Let $\gamma$ be a degree zero cocycle in $\mathcal{F}_{n-1}\mathsf{dfGC}$ and $\xi$ be a degree zero vector in $\mathcal{F}^{\mathfrak{c}}_{m-1} \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$ satisfying conditions* (5.13), (5.14). *If equation* (8.2) *holds and*

$$2 \le m \le n-1\,, \tag{8.9}$$

*then*

$$\partial^{\mathrm{Hoch}} \xi_{m,k} = 0 \qquad \forall \;\; k \ge 0\,, \tag{8.10}$$

*where $\partial^{\mathrm{Hoch}}$ is defined in* (7.20).

Proof. Since $\gamma \in \mathcal{F}_{n-1}\mathsf{dfGC}$, inequality (8.9) implies that

$$\exp\big(\mathrm{ad}_{J(\gamma)}\big)\alpha\,(\mathbf{s}^{-1}\, \mathbf{t}^{\mathfrak{o}}_{m,k}) = \alpha\,(\mathbf{s}^{-1}\, \mathbf{t}^{\mathfrak{o}}_{m,k})\,.$$

Next, using the inequality $m \ge 2$ and inclusion (8.5), it is not hard to show that

$$\mathrm{ad}^q_{\xi}(\alpha)\,(\mathbf{s}^{-1}\, \mathbf{t}^{\mathfrak{o}}_{m,k}) = 0 \qquad \forall \;\; q \ge 2\,.$$

Thus, we conclude that

$$[\xi, \alpha](\mathbf{s}^{-1}\, \mathbf{t}^{\mathfrak{o}}_{m,k}) = 0\,. \tag{8.11}$$

Using inclusion (8.5) once again we can simplify the left hand side of (8.11). Namely,

$$[\xi, \alpha](\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k}) = \sum_{p=1}^{k-1} (-1)^{p-1} \xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k-1}) \circ_{\mathfrak{o},p} \alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{2})$$

$$+(-1)^{k-1}\alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{2}) \circ_{\mathfrak{o},1} \xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k-1}) - \alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{2}) \circ_{\mathfrak{o},2} \xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k-1}) =$$

$$\sum_{p=1}^{k-1} (-1)^{p-1} \xi_{m,k-1} \circ_{\mathfrak{o},p} \Gamma_{\circ\circ}$$

$$+(-1)^{k-1}\Gamma_{\circ\circ} \circ_{\mathfrak{o},1} \xi_{m,k-1} - \Gamma_{\circ\circ} \circ_{\mathfrak{o},2} \xi_{m,k-1}\,.$$

Hence

$$[\xi, \alpha](\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k}) = -\partial^{\mathrm{Hoch}} \xi_{m,k-1}\,,$$

where $\partial^{\mathrm{Hoch}}$ is defined in (7.20).

Thus (8.11) implies that

$$\partial^{\mathrm{Hoch}} \xi_{m,k-1} = 0$$

for all $k \ge 1$. The desired statement is proved. □

Combining Claim 8.1 with Corollary A.2 we deduce the following.

**Corollary 8.1** *If conditions of Claim 8.1 hold then the white vertex in each graph in the linear combination $\xi_{m,1}$ has valency $1$.* □

## 8.2 Pikes in $\xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,0})$ can "be killed"

Let $\psi$ be a degree $-1$ vector in

$$\mathcal{F}_1 \mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$$

Since $\mathrm{ad}_{[\psi,\alpha]}$ acts trivially on $\alpha$, we have

$$\exp\left(\mathrm{ad}_{[\psi,\alpha]}\right)(\alpha) = \alpha$$

and hence

$$\exp\left(\mathrm{ad}_{\mathrm{CH}(\xi,[\psi,\alpha])}\right)(\alpha) = \exp(\mathrm{ad}_{\xi})(\alpha)\,. \tag{8.12}$$

Thus equation (8.2) will still hold after replacing $\xi$ by $\mathrm{CH}(\xi, [\psi, \alpha])$.

We claim that

**Claim 8.2** *If inclusion* (8.5) *holds, then there exists a degree $-1$ vector*

$$\psi \in \mathcal{F}_1 \mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$$

*such that*

$$\mathrm{CH}(\xi, [\psi, \alpha]) \in \mathcal{F}^{\mathfrak{c}}_{m-1} \mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra})\,. \tag{8.13}$$

*and each graph in the linear combination*

$$\mathrm{CH}(\xi, [\psi, \alpha])(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,0})$$

*does not have pikes.*

Proof. Since the graphs in $\xi_{m,0}$ do not have white vertices, the vector $\xi_{m,0}$ is cocycle in the complex (B.1) with the differential $\mathfrak{d}$ (B.5) (see Appendix B).

Let us denote by $\xi^r_{m,0}$ the linear combination in $\mathsf{KGra}(m,0)^{\mathfrak{o}}$ which is obtained from $\xi_{m,0}$ by retaining only the graphs with exactly $r$ pikes. According to Lemma B.3 from Appendix B we have

$$\mathfrak{d}\mathfrak{d}^*(\xi^r_{m,0}) = r\xi^r_{m,0}\,.$$

Thus, if

$$\psi_{m-1,1} = -\sum_{r\ge 1} \frac{1}{r}\mathfrak{d}^*(\xi^r_{m,0}), \tag{8.14}$$

then each graph in the linear combination

$$\xi_{m,0} + \mathfrak{d}(\psi_{n-1,1}) \tag{8.15}$$

does not have pikes.

Next, we define a degree $-1$ vector

$$\psi \in \mathcal{F}_1 \mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra})$$

by setting

$$\psi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m-1,1}) = \psi_{m-1,1}\,, \qquad \psi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{c}}_{n_1}) = \psi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{k_1}) = \psi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n_2,k_2}) = 0 \tag{8.16}$$

for all $n_1, k_1 \ge 2$ and for all pairs $(n_2,k_2) \neq (m-1,1)$.

Then we consider the vector

$$\xi' = \mathrm{CH}(\xi, [\psi,\alpha])\,. \tag{8.17}$$

Using the fact that $\psi_{m-1,1}$ belongs to the kernel of the differential $\partial^{\mathrm{Hoch}}$ (7.20), it is not hard to show that

$$[\psi,\alpha] \in \mathcal{F}^{\mathfrak{c}}_{m-1}\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})\,. \tag{8.18}$$

Hence,

$$\mathrm{CH}(\xi,[\psi,\alpha])\,(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,0}) = \xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,0}) + [\psi,\alpha](\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,0}) =$$

$$\xi_{m,0} + \sum_{i=1}^{m} (\tau_{m,i}, \mathrm{id})\Big(\psi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m-1,1}) \circ_{\mathfrak{o},1} \alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{1,0})\Big) =$$

$$= \xi_{m,0} + \sum_{i=1}^{m} (\tau_{m,i}, \mathrm{id})\Big(\psi_{m-1,1} \circ_{\mathfrak{o},1} \Gamma^{\mathrm{br}}_0\Big)$$

where $\tau_{m,i}$ is defined in (B.2).

Thus, by definition of the operator $\mathfrak{d}$ (B.5), we get

$$\mathrm{CH}(\xi,[\psi,\alpha])\,(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,0}) = \xi_{m,0} + \mathfrak{d}\psi_{m-1,1}\,.$$

Since each graph the linear combination $\xi_{m,0} + \mathfrak{d}\psi_{m-1,1}$ does not have pikes, the claim is proved. $\square$

### 8.3 The case $m = n-1$

If $m = n-1$, then applying both sides of (8.2) to $\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,k}$ and using inclusions (8.3), (8.5) together with the inequality $n \geq 3$, we deduce that

$$[J(\gamma), \alpha](\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,0}) = [\xi, \alpha](\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,0})\,. \tag{8.19}$$

Since $J(\gamma) \in \mathcal{F}^{\mathfrak{c}}_{n-1}\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$,

$$\begin{aligned}[J(\gamma), \alpha](\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,0}) &= -\alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{1,0}) \circ_{1,\mathfrak{c}} J(\gamma)(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{c}}_{n}) = \\ &= -\Gamma^{\mathrm{br}}_0 \circ_{1,\mathfrak{c}} \gamma(1_n) = -\gamma(1_n)\,.\end{aligned} \tag{8.20}$$

For $[\xi, \alpha](\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,0})$ we get

$$\begin{aligned}[\xi, \alpha](\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,0}) = &- \sum_{\tau \in \mathrm{Sh}_{2,n-2}} (\tau, \mathrm{id})\Big(\xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n-1,0}) \circ_{1,\mathfrak{c}} \alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{c}}_{2})\Big) \\ &+ \sum_{i=1}^{n} (\tau_{n,i}, \mathrm{id})\Big(\xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n-1,1}) \circ_{1,\mathfrak{o}} \alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{1,0})\Big) \\ &- \sum_{i=1}^{n} (\sigma_{n,i}, \mathrm{id})\Big(\alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{1,1}) \circ_{1,\mathfrak{o}} \xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n-1,0})\Big) = \\ &- \sum_{\tau \in \mathrm{Sh}_{2,n-2}} (\tau, \mathrm{id})\Big(\xi_{n-1,0} \circ_{1,\mathfrak{c}} \Gamma_{\bullet\!-\!\bullet}\Big) \\ &+ \sum_{i=1}^{n} (\tau_{n,i}, \mathrm{id})\Big(\xi_{n-1,1} \circ_{1,\mathfrak{o}} \Gamma^{\mathrm{br}}_0\Big) - \sum_{i=1}^{n} (\sigma_{n,i}, \mathrm{id})\Big(\Gamma^{\mathrm{br}}_1 \circ_{1,\mathfrak{o}} \xi_{n-1,0}\Big)\,,\end{aligned}$$

where $\tau_{n,i}$ and $\sigma_{n,i}$ are families of cycles in $S_n$ defined in (B.2) and (B.3), respectively.

Thus

$$\begin{aligned}[\xi, \alpha](\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,0}) = &- \sum_{\tau \in \mathrm{Sh}_{2,n-2}} (\tau, \mathrm{id})\Big(\xi_{n-1,0} \circ_{1,\mathfrak{c}} \Gamma_{\bullet\!-\!\bullet}\Big) \\ &+ \sum_{i=1}^{n} (\tau_{n,i}, \mathrm{id})\Big(\xi_{n-1,1} \circ_{1,\mathfrak{o}} \Gamma^{\mathrm{br}}_0\Big) - \sum_{i=1}^{n} (\sigma_{n,i}, \mathrm{id})\Big(\Gamma^{\mathrm{br}}_1 \circ_{1,\mathfrak{o}} \xi_{n-1,0}\Big)\,.\end{aligned} \tag{8.21}$$

Corollary 8.1 implies that the white vertex in each graph of $\xi_{n-1,1}$ has valency $1$. Therefore, all graphs in the sum

$$\sum_{i=1}^{n} (\tau_{n,i}, \mathrm{id})\Big(\xi_{n-1,1} \circ_{1,\mathfrak{o}} \Gamma^{\mathrm{br}}_0\Big) \tag{8.22}$$

have pikes.

On the other hand, all graphs in $\xi_{n-1,0}$ do not have pikes due to Claim 8.2 and graphs in $\gamma$ do not have pikes either according to Condition 8.1.

Thus equation (8.19) implies that the sum (8.22) equals to the linear combination of all graphs with pikes from the expression

$$\sum_{\tau \in \mathrm{Sh}_{2,n-2}} (\tau, \mathrm{id})\Big(\xi_{n-1,0} \circ_{1,\mathfrak{c}} \Gamma_{\bullet\!-\!\bullet}\Big)\,. \tag{8.23}$$

In other words[11],

$$\sum_{i=1}^{n} (\tau_{n,i}, \mathrm{id}) \Big( \xi_{n-1,1} \circ_{1,\mathfrak{o}} \Gamma^{\mathrm{br}}_{0} \Big) = - \sum_{i=1}^{n} (\tau_{n,i}, \mathrm{id}) \Big( \Gamma^{>}_{\bullet\!-\!\bullet} \circ_{1,\mathfrak{c}} \xi_{n-1,0} \Big) \,, \tag{8.24}$$

where

$$\Gamma^{>}_{\bullet\!-\!\bullet} = \overset{1}{\bullet}\!\longrightarrow\!\overset{2}{\bullet} \,. \tag{8.25}$$

In addition we observe that the last sum in (8.21) can be rewritten as

$$\sum_{i=1}^{n} (\sigma_{n,i}, \mathrm{id}) \Big( \Gamma^{\mathrm{br}}_{1} \circ_{1,\mathfrak{o}} \xi_{n-1,0} \Big) = \sum_{i=1}^{n} (\tau_{n,i}, \mathrm{id}) \Big( \Gamma^{<}_{\bullet\!-\!\bullet} \circ_{1,\mathfrak{c}} \xi_{n-1,0} \Big) \,, \tag{8.26}$$

where

$$\Gamma^{<}_{\bullet\!-\!\bullet} = \overset{1}{\bullet}\!\longleftarrow\!\overset{2}{\bullet} \,. \tag{8.27}$$

Combining equation (8.20), (8.21), and (8.24) with (8.26) we deduce that

$$\gamma(1_n) = \sum_{\tau \in \mathrm{Sh}_{2,n-2}} (\tau, \mathrm{id}) \Big( \xi_{n-1,0} \circ_{1,\mathfrak{c}} \Gamma_{\bullet\!-\!\bullet} \Big) \tag{8.28}$$

$$+ \sum_{i=1}^{n} (\tau_{n,i}, \mathrm{id}) \Big( \Gamma^{>}_{\bullet\!-\!\bullet} \circ_{1,\mathfrak{c}} \xi_{n-1,0} \Big) + \sum_{i=1}^{n} (\tau_{n,i}, \mathrm{id}) \Big( \Gamma^{<}_{\bullet\!-\!\bullet} \circ_{1,\mathfrak{c}} \xi_{n-1,0} \Big) = [\, \Gamma_{\bullet\!-\!\bullet} \,,\, \xi_{n-1,0} \,]$$

Thus $\gamma(1_n)$ is exact and it is a contradiction.

The remaining case $m < n-1$ is more involved and the work on this case occupies Subsections 8.4, 8.5.

## 8.4 The vector $\xi(\mathbf{t}^{\mathfrak{o}}_{m,0})$ is a cocycle in $\mathsf{dfGC}$

Let us now go back to equation (8.2) with

$$\xi \in \mathcal{F}^{\mathfrak{c}}_{m-1} \mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra}) \,,$$

$$\gamma \in \mathcal{F}_{n-1} \mathsf{dfGC} \,,$$

and $n > m+1$ .

Due to Claim 8.2 we may assume, without loss of generality, that each graph in the linear combination

$$\xi_{m,0} = \xi(\mathbf{s}^{-1}\, \mathbf{t}^{\mathfrak{o}}_{m,0}) \tag{8.29}$$

does not have pikes.

Let us now prove that

**Claim 8.3** *Under the above assumptions, the vector* (8.29) *is a degree* $-1$ *cocycle in* $\mathsf{dfGC}$ .

[11] Here we use the fact that $\xi_{n-1,0}$ carries an odd degree.

Proof. Since $\gamma \in \mathcal{F}_{n-1}\mathsf{dfGC}$ for $n > m+1$, equation (8.2) implies that

$$[\xi, \alpha](\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m+1,0}) = 0\,. \tag{8.30}$$

On the other hand,

$$[\xi, \alpha](\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m+1,0}) = -\sum_{\tau\in \mathrm{Sh}_{2,m-1}} (\tau, \mathrm{id})\Big(\xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,0}) \circ_{\mathfrak{c},1} \alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{c}}_{2})\Big)$$

$$+\sum_{i=1}^{m+1}(\tau_{m+1,i}, \mathrm{id})\Big(\xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,1}) \circ_{\mathfrak{o},1} \alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{1,0})\Big)$$

$$-\sum_{i=1}^{m+1}(\sigma_{m+1,i}, \mathrm{id})\Big(\alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{1,1}) \circ_{\mathfrak{o},1} \xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,0})\Big) =$$

$$= -\sum_{\tau\in \mathrm{Sh}_{2,m-1}} (\tau, \mathrm{id})\Big(\xi_{m,0} \circ_{\mathfrak{c},1} \Gamma_{\bullet\!-\!\bullet}\Big) + \sum_{i=1}^{m+1}(\tau_{m+1,i}, \mathrm{id})\Big(\xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,1}) \circ_{\mathfrak{o},1} \Gamma^{\mathrm{br}}_{0}\Big)$$

$$-\sum_{i=1}^{m+1}(\sigma_{m+1,i}, \mathrm{id})\Big(\Gamma^{\mathrm{br}}_{1} \circ_{\mathfrak{o},1} \xi_{m,0}\Big)\,,$$

where $\tau_{m+1,i}$ and $\sigma_{m+1,i}$ are families cycles defined in (B.2) and (B.3), respectively.

Thus

$$-\sum_{\tau\in \mathrm{Sh}_{2,m-1}} (\tau, \mathrm{id})\Big(\xi_{m,0} \circ_{\mathfrak{c},1} \Gamma_{\bullet\!-\!\bullet}\Big) + \sum_{i=1}^{m+1}(\tau_{m+1,i}, \mathrm{id})\Big(\xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,1}) \circ_{\mathfrak{o},1} \Gamma^{\mathrm{br}}_{0}\Big) \tag{8.31}$$

$$-\sum_{i=1}^{m+1}(\sigma_{m+1,i}, \mathrm{id})\Big(\Gamma^{\mathrm{br}}_{1} \circ_{\mathfrak{o},1} \xi_{m,0}\Big) = 0\,.$$

Since graphs in $\xi_{m,0}$ do not have pikes and each graph in the sum

$$\sum_{i=1}^{m+1}(\tau_{m+1,i}, \mathrm{id})\Big(\xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,1}) \circ_{\mathfrak{o},1} \Gamma^{\mathrm{br}}_{0}\Big) \tag{8.32}$$

has a pike, we conclude that (8.32) equals to the linear combination formed by all graphs with pikes in the sum

$$\sum_{\tau\in \mathrm{Sh}_{2,m-1}} (\tau, \mathrm{id})\Big(\xi_{m,0} \circ_{\mathfrak{c},1} \Gamma_{\bullet\!-\!\bullet}\Big)\,. \tag{8.33}$$

It is not hard to see that this linear combination equals[12]

$$-\sum_{i=1}^{m+1}(\tau_{m+1,i}, \mathrm{id})\Big(\Gamma^{>}_{\bullet\!-\!\bullet} \circ_{1,\mathfrak{c}} \xi_{m,0}\Big)\,,$$

where $\Gamma^{>}_{\bullet\!-\!\bullet}$ is defined in equation (8.25).

[12] Here we use the fact that $\xi_{m,0}$ carries an odd degree.

Therefore,

$$\sum_{i=1}^{m+1}(\tau_{m+1,i},\mathrm{id})\Big(\xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,1})\circ_{\mathfrak{o},1}\Gamma^{\mathrm{br}}_{0}\Big) = -\sum_{i=1}^{m+1}(\tau_{m+1,i},\mathrm{id})\Big(\Gamma^{>}_{\bullet\!-\!\bullet}\circ_{1,\mathfrak{c}}\xi_{m,0}\Big)\,. \tag{8.34}$$

Let us also observe that the last sum in (8.31) can be rewritten as

$$\sum_{i=1}^{m+1}(\sigma_{m+1,i},\mathrm{id})\Big(\Gamma^{\mathrm{br}}_{1}\circ_{\mathfrak{o},1}\xi_{m,0}\Big) = \sum_{i=1}^{m+1}(\tau_{m+1,i},\mathrm{id})\Big(\Gamma^{<}_{\bullet\!-\!\bullet}\circ_{1,\mathfrak{c}}\xi_{m,0}\Big)\,, \tag{8.35}$$

where $\Gamma^{<}_{\bullet\!-\!\bullet}$ is defined in equation (8.27).

Thus, combining (8.31), (8.34), and (8.35) we get

$$\sum_{\tau\in\mathrm{Sh}_{2,m-1}}(\tau,\mathrm{id})\Big(\xi_{m,0}\circ_{\mathfrak{c},1}\Gamma_{\bullet\!-\!\bullet}\Big) + \sum_{i=1}^{m+1}(\tau_{m+1,i},\mathrm{id})\Big(\Gamma^{>}_{\bullet\!-\!\bullet}\circ_{1,\mathfrak{c}}\xi_{m,0}\Big) \tag{8.36}$$
$$+\sum_{i=1}^{m+1}(\tau_{m+1,i},\mathrm{id})\Big(\Gamma^{<}_{\bullet\!-\!\bullet}\circ_{1,\mathfrak{c}}\xi_{m,0}\Big) = 0\,.$$

On the other hand, the left hand side of (8.36) equals $[\Gamma_{\bullet\!-\!\bullet},\xi_{m,0}]$. Claim 8.3 is proved.

□

Due to the above claim, the vector $\xi_{m,0}$ is a degree $-1$ cocycle with respect to the differential $[\Gamma_{\bullet\!-\!\bullet},\ ]$, if we view $\xi_{m,0}$ as a vector in $\mathsf{dfGC}$.

Since $m\geq 2$, we also know that $\xi_{m,0}\in\mathcal{F}_1\mathsf{dfGC}$. On the other hand each degree $-1$ cocycle in $\mathcal{F}_1\mathsf{dfGC}$ is trivial. Thus we conclude that, there exists a degree $-2$ vector

$$\psi_{m-1}\in\mathsf{dfGC} \tag{8.37}$$

such that

$$\xi_{m,0} = [\Gamma_{\bullet\!-\!\bullet},\psi_{m-1}]\,. \tag{8.38}$$

Using $\psi_{m-1}$ we define a vector

$$\psi\in\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ},\mathsf{KGra}) \tag{8.39}$$

by setting

$$\psi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m-1,0}) = \psi_{m-1}\,,\qquad \psi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m-1,k}) = \Gamma^{\mathrm{br}}_{k}\circ_{1,\mathfrak{c}}\psi_{m-1}\,, \tag{8.40}$$
$$\psi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m_1,k_1}) = \psi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{k_2}) = \psi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{c}}_{m_2}) = 0$$

for all $m_2,k_2\geq 2$, $m_1\neq m-1$, $k_1\geq 0$, and $k\geq 1$.

Let us now consider the degree zero vector

$$\xi' = \mathrm{CH}(\xi,[\psi,\alpha])\in\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ},\mathsf{KGra})\,. \tag{8.41}$$

The vector $\xi'$ satisfies condition (5.13) since so does the vector $\xi$. Moreover, using (5.14) and (5.17) it is easy to see that $\xi'$ satisfies condition (5.14) as well.

Next, using the inclusions $\psi\in\mathcal{F}^{\mathfrak{c}}_{m-2}\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ},\mathsf{KGra})$, $\xi\in\mathcal{F}_{m-1}\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ},\mathsf{KGra})$ together with the obvious identity

$$\partial^{\mathrm{Hoch}}\big(\Gamma^{\mathrm{br}}_{k}\circ_{1,\mathfrak{c}}\psi_{m-1}\big) = 0\,,\qquad \forall\ \ k\geq 0$$

we conclude that

$$\xi' \in \mathcal{F}_{m-1}\mathrm{Conv}(\mathfrak{oc}_{\circ}^{\vee}, \mathsf{KGra})\,. \tag{8.42}$$

Furthermore,

$$\xi'(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,0}) = \xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,0}) + [\psi,\alpha](\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,0}) =$$

$$\xi_{m,0} \;-\; \sum_{\tau\in \mathrm{Sh}_{2,m-2}} (\tau,\mathrm{id})\,\Big(\psi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m-1,0}) \circ_{1,\mathfrak{c}} \alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{c}}_{2})\Big)$$

$$+\sum_{i=1}^{m}(\tau_{m,i},\mathrm{id})\Big(\psi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m-1,1}) \circ_{1,\mathfrak{o}} \alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{1,0})\Big)$$

$$+\sum_{i=1}^{m}(\sigma_{m,i},\mathrm{id})\Big(\alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{1,1}) \circ_{1,\mathfrak{o}} \psi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m-1,0})\Big) =$$

$$\xi_{m,0} \;-\; \sum_{\tau\in \mathrm{Sh}_{2,m-2}} (\tau,\mathrm{id})\,\Big(\psi_{m-1} \circ_{1,\mathfrak{c}} \Gamma_{\bullet\!-\!\bullet}\Big)$$

$$+\sum_{i=1}^{m}(\tau_{m,i},\mathrm{id})\Big(\big(\Gamma^{\mathrm{br}}_{1} \circ_{1,\mathfrak{c}} \psi_{m-1}\big) \circ_{1,\mathfrak{o}} \Gamma^{\mathrm{br}}_{0}\Big) + \sum_{i=1}^{m}(\sigma_{m,i},\mathrm{id})\Big(\Gamma^{\mathrm{br}}_{1} \circ_{1,\mathfrak{o}} \psi_{m-1}\Big)\,,$$

where $\tau_{m,i}$ and $\sigma_{m,i}$ are the families of cycles in $S_m$ defined in (B.2), (B.3), respectively.

Thus

$$\xi'(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,0}) = \xi_{m,0} \;-\; \sum_{\tau\in \mathrm{Sh}_{2,m-2}} (\tau,\mathrm{id})\,\Big(\psi_{m-1} \circ_{1,\mathfrak{c}} \Gamma_{\bullet\!-\!\bullet}\Big) \tag{8.43}$$

$$+\sum_{i=1}^{m}(\tau_{m,i},\mathrm{id})\Big(\big(\Gamma^{\mathrm{br}}_{1} \circ_{1,\mathfrak{c}} \psi_{m-1}\big) \circ_{1,\mathfrak{o}} \Gamma^{\mathrm{br}}_{0}\Big) + \sum_{i=1}^{m}(\sigma_{m,i},\mathrm{id})\Big(\Gamma^{\mathrm{br}}_{1} \circ_{1,\mathfrak{o}} \psi_{m-1}\Big)\,.$$

The last two sums in the right hand side of equation (8.43) can be simplified as follows[13]

$$\sum_{i=1}^{m}(\tau_{m,i},\mathrm{id})\Big(\big(\Gamma^{\mathrm{br}}_{1} \circ_{1,\mathfrak{c}} \psi_{m-1}\big) \circ_{1,\mathfrak{o}} \Gamma^{\mathrm{br}}_{0}\Big) = \sum_{i=1}^{m}(\tau_{m,i},\mathrm{id})\Big(\Gamma^{>}_{\bullet\!-\!\bullet} \circ_{1,\mathfrak{c}} \psi_{m-1}\Big)\,, \tag{8.44}$$

$$\sum_{i=1}^{m}(\sigma_{m,i},\mathrm{id})\Big(\Gamma^{\mathrm{br}}_{1} \circ_{1,\mathfrak{o}} \psi_{m-1}\Big) = \sum_{i=1}^{m}(\tau_{m,i},\mathrm{id})\Big(\Gamma^{<}_{\bullet\!-\!\bullet} \circ_{1,\mathfrak{c}} \psi_{m-1}\Big)\,, \tag{8.45}$$

where the graphs $\Gamma^{>}_{\bullet\!-\!\bullet}$ and $\Gamma^{<}_{\bullet\!-\!\bullet}$ are defined in (8.25) and (8.27), respectively.

Combining (8.38), (8.43), (8.44), and (8.45), we get

$$\xi'(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,0}) = \xi_{m,0} \;-\; \sum_{\tau\in \mathrm{Sh}_{2,m-2}} (\tau,\mathrm{id})\,\Big(\psi_{m-1} \circ_{1,\mathfrak{c}} \Gamma_{\bullet\!-\!\bullet}\Big) \tag{8.46}$$

$$\sum_{i=1}^{m}(\tau_{m,i},\mathrm{id})\Big(\Gamma^{>}_{\bullet\!-\!\bullet} \circ_{1,\mathfrak{c}} \psi_{m-1}\Big) + \sum_{i=1}^{m}(\tau_{m,i},\mathrm{id})\Big(\Gamma^{<}_{\bullet\!-\!\bullet} \circ_{1,\mathfrak{c}} \psi_{m-1}\Big) =$$

$$\xi_{m,0} - [\Gamma_{\bullet\!-\!\bullet}, \psi_{m-1}] = 0\,.$$

Using Claims 8.2, 8.3 together with the above computations, we conclude the following.

[13] Here we use the fact that $\psi_{m-1}$ carries an even degree.

**Claim 8.4** *Let $\gamma$ be a degree zero cocycle in $\mathcal{F}_{n-1}\mathsf{dfGC}$ and $\xi$ be a degree zero vector in $\mathcal{F}^{\mathfrak{c}}_{m-1}\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$ satisfying conditions* (5.13), (5.14). *If equation* (8.2) *holds and*

$$2 \le m < n-1\,, \tag{8.47}$$

*then there exists a degree zero vector*

$$\xi' \in \mathcal{F}^{\mathfrak{c}}_{m-1}\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$$

*satisfying conditions* (5.13), (5.14), *and such that*

$$\exp\big(\mathrm{ad}_{J(\gamma)}\big)\alpha = \exp\big(\mathrm{ad}_{\xi'}\big)\alpha\,,$$

$$\xi'(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,0}) = 0\,.$$

□

## 8.5 If $\xi_{m,0} = 0$ then the remaining vectors $\xi_{m,k}$ can be "killed" by adjusting $\xi$

The following claim completes the proof of faithfulness of the action of $\exp\big(H^0(\mathsf{dfGC})\big)$ on homotopy classes of stable formality quasi-isomorphism. Hence it also completes the proof of Theorem 6.2.

**Claim 8.5** *Let $\gamma$ be a degree zero cocycle in $\mathcal{F}_{n-1}\mathsf{dfGC}$ and $\xi$ be a degree zero vector in $\mathcal{F}^{\mathfrak{c}}_{m-1}\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$ satisfying conditions* (5.13), (5.14). *If equation* (8.2) *holds and*

$$2 \le m < n-1\,, \tag{8.48}$$

*then there exists a degree zero vector*

$$\xi' \in \mathcal{F}^{\mathfrak{c}}_{m}\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$$

*satisfying conditions* (5.13), (5.14) *and such that*

$$\exp\big(\mathrm{ad}_{J(\gamma)}\big)\alpha = \exp\big(\mathrm{ad}_{\xi'}\big)\alpha\,. \tag{8.49}$$

Proof. Using the results of the previous subsection we may assume, without loss of generality, that the vector $\xi$ in (8.2) satisfies the condition

$$\xi(\mathbf{t}^{\mathfrak{o}}_{m,0}) = 0\,. \tag{8.50}$$

Suppose that $\xi_{m,k} \neq 0$ for some $k \ge 1$. Hence there exists an integer $k' \ge 1$ such that

$$\xi_{m,k} = 0 \qquad \forall \;\; k < k'\,. \tag{8.51}$$

We will construct a degree $-1$ vector

$$\psi_{k'} \in \mathcal{F}_{m+k'-2}\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}) \tag{8.52}$$

such that

$$\mathrm{CH}(\xi, [\psi_{k'}, \alpha]) \in \mathcal{F}^{\mathfrak{c}}_{m-1}\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}) \tag{8.53}$$

and

$$\mathrm{CH}(\xi, [\psi_{k'}, \alpha])(\mathbf{t}^{\mathfrak{o}}_{m,k}) = 0 \qquad \forall \;\; k \le k'\,. \tag{8.54}$$

The construction consists of two steps. In the first step we prove that there exists a degree $-1$ vector

$$\psi^{(1)} \in \mathcal{F}_{m+k'-2}\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra}) \tag{8.55}$$

such that

$$\mathrm{CH}(\xi, [\psi^{(1)}, \alpha]) \in \mathcal{F}^{\mathfrak{c}}_{m-1}\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra})\,, \tag{8.56}$$

$$\mathrm{CH}(\xi, [\psi^{(1)}, \alpha])(\mathbf{t}^{\mathfrak{o}}_{m,k}) = 0 \tag{8.57}$$

for all $k < k'$, and the vector

$$\mathrm{CH}(\xi, [\psi^{(1)}, \alpha])(\mathbf{t}^{\mathfrak{o}}_{m,k'}) \tag{8.58}$$

satisfies Properties A.1, A.2, i.e. for each graph in (8.58) white vertices have valency 1 and the linear combination (8.58) is anti-symmetric with respect to permutations of labels on white vertices. Let us set

$$\xi^{(1)} = \mathrm{CH}(\xi, [\psi^{(1)}, \alpha])\,.$$

In the second step, we prove that that there exists a degree $-1$ vector

$$\psi^{(2)} \in \mathcal{F}_{m+k'-2}\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra}) \tag{8.59}$$

such that

$$\mathrm{CH}(\xi^{(1)}, [\psi^{(2)}, \alpha]) \in \mathcal{F}^{\mathfrak{c}}_{m-1}\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra})\,, \tag{8.60}$$

and

$$\mathrm{CH}(\xi^{(1)}, [\psi^{(2)}, \alpha])(\mathbf{t}^{\mathfrak{o}}_{m,k}) = 0 \tag{8.61}$$

for all $k \le k'$.

Then we would have

$$\mathrm{CH}(\xi^{(1)}, [\psi^{(2)}, \alpha]) = \mathrm{CH}\big(\mathrm{CH}(\xi, [\psi^{(1)}, \alpha]), [\psi^{(2)}, \alpha]\big) =$$

$$\mathrm{CH}\big(\xi, \mathrm{CH}([\psi^{(1)}, \alpha], [\psi^{(2)}, \alpha])\big)\,.$$

Finally, rewriting $\mathrm{CH}([\psi^{(1)}, \alpha], [\psi^{(2)}, \alpha])$ in the form

$$\mathrm{CH}([\psi^{(1)}, \alpha], [\psi^{(2)}, \alpha]) = [\psi_{k'}, \alpha]$$

we would get a vector (8.52) with the desired property (8.54).

**Step 1.** If $k' = 1$ then the linear combination $\xi_{m,k'}$ already satisfies Properties A.1, A.2 due to Corollary 8.1. So, in this case, we proceed to Step 2. In Step 1, it remains to consider the case $k' \ge 2$.

Due to Claim 8.1 the vector

$$\xi_{m,k'} \in \mathbf{s}^{2m-2+k'}\Big(\mathsf{KGra}(m,k')^{\mathfrak{o}}\Big)^{S_m}$$

is a cocycle in the complex (A.1) with the differential (A.2). Thus, Corollary A.1 implies that there exists a vector

$$\psi_{m,k'-1} \in \mathbf{s}^{2m-3+k'}\Big(\mathsf{KGra}(m,k'-1)^{\mathfrak{o}}\Big)^{S_m} \tag{8.62}$$

such that the difference

$$\xi_{m,k'} + \partial^{\mathrm{Hoch}}\psi_{m,k'-1} \tag{8.63}$$

satisfies Properties A.1, A.2.

So we form a degree $-1$ vector (8.55) by setting

$$\psi^{(1)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k'-1}) = \psi_{m,k'-1}\,, \qquad \psi^{(1)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{c}}_{m_1}) = \psi^{(1)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{k_1}) = \psi^{(1)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m_2,k_2}) = 0 \tag{8.64}$$

for all $m_1, k_1 \ge 2$ and $(m_2, k_2) \neq (m, k'-1)$.

Let us consider the vector

$$\xi^{(1)} = \mathrm{CH}(\xi, [\psi^{(1)}, \alpha])\,. \tag{8.65}$$

Since

$$\psi^{(1)} \in \mathcal{F}^{\mathfrak{c}}_{m-1}\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra}) \tag{8.66}$$

we have

$$\xi^{(1)} \in \mathcal{F}^{\mathfrak{c}}_{m-1}\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra})\,. \tag{8.67}$$

Furthermore, since $m \ge 2$ and hence $qm - q \ge m$ for all $q \ge 2$ we have

$$\mathrm{CH}(\xi, [\psi^{(1)}, \alpha])\,(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k}) = \xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k}) + [\psi^{(1)}, \alpha](\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k}) = \tag{8.68}$$

$$\xi_{m,k} + [\psi^{(1)}, \alpha](\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k})\,.$$

Next, we have

$$[\psi^{(1)}, \alpha](\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k}) = 0 \qquad \forall \;\; k < k'-1$$

because

$$\psi^{(1)} \in \mathcal{F}_{m+k'-2}\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra}) \tag{8.69}$$

and a direct computation show that $[\psi^{(1)}, \alpha](\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k'-1}) = 0$. Thus

$$[\psi^{(1)}, \alpha](\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k}) = 0 \qquad \forall \;\; k < k' \tag{8.70}$$

and equation (8.57) indeed holds for all $k < k'$.

Finally, a direct computation shows that

$$[\psi^{(1)}, \alpha](\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k'}) = \partial^{\mathrm{Hoch}}\psi_{m,k'-1}\,. \tag{8.71}$$

Thus,

$$\mathrm{CH}(\xi, [\psi^{(1)}, \alpha])\,(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k'}) = \xi_{m,k'} + \partial^{\mathrm{Hoch}}\psi_{m,k'-1}$$

and this vector satisfies Properties A.1, A.2 by construction.

Step 1 is completed.

**Step 2.**

Using the inclusions $J(\gamma) \in \mathcal{F}^{\mathfrak{c}}_{n-1}\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra})$, (8.67), the inequality $n-1 > m$, and the equation $\xi^{(1)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{c}}_{n_1}) = 0$ it is not hard to see that equation

$$\exp\big(\mathrm{ad}_{J(\gamma)}\big)\alpha = \exp\big(\mathrm{ad}_{\xi^{(1)}}\big)\alpha\,.$$

implies the identity

$$[\xi^{(1)}, \alpha](\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m+1,k'-1}) = 0\,. \tag{8.72}$$

A direct computation shows that

$$[\xi^{(1)}, \alpha](\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m+1,k'-1}) = -\partial^{\mathrm{Hoch}}\big(\xi^{(1)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m+1,k'-2})\big) + \mathfrak{d}\big(\xi^{(1)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k'})\big)\,,$$

where $\partial^{\mathrm{Hoch}}$ is defined in (A.2) and $\mathfrak{d}$ is defined in (B.5).

Thus

$$\mathfrak{d}\big(\xi^{(1)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k'})\big) = \partial^{\mathrm{Hoch}}\big(\xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m+1,k'-2})\big)\,. \tag{8.73}$$

On the other hand, the vector $\mathfrak{d}\big(\xi^{(1)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k'})\big)$ satisfies Properties A.1, A.2. Therefore, the second statement of Corollary A.1 implies that

$$\mathfrak{d}\big(\xi^{(1)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k'})\big) = 0\,. \tag{8.74}$$

Furthermore, since $k' \ge 1$, the vector $\xi^{(1)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k'})$ is $\mathfrak{d}$-exact due to Lemma B.3. In other words, there exists a vector

$$\psi_{m-1,k'+1} \in \mathbf{s}^{2m-3+k'}\Big(\mathsf{KGra}(m-1,k'+1)\Big)^{S_{m-1}} \tag{8.75}$$

which satisfies Properties A.1, A.2 and such that

$$\xi^{(1)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k'}) = -\mathfrak{d}(\psi_{m-1,k'+1})\,. \tag{8.76}$$

Using $\psi_{m-1,k'+1}$ we define a degree $-1$ vector $\psi^{(2)} \in \mathrm{Conv}(\mathfrak{oc}^{\vee},\mathsf{KGra})$ by setting

$$\psi^{(2)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m-1,k'+1}) = \psi_{m-1,k'+1}\,, \qquad \psi^{(2)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{c}}_{m_1}) = 0\,, \tag{8.77}$$

$$\psi^{(2)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{k_1}) = \psi^{(2)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m_2,k_2}) = 0$$

for all $m_1, k_1 \ge 2$ and $(m_2,k_2) \neq (m-1,k'+1)$.

Let us consider the vector

$$\xi^{(2)} = \mathrm{CH}(\xi^{(1)}, [\psi^{(2)},\alpha])\,. \tag{8.78}$$

Using the inclusion

$$\psi^{(2)} \in \mathcal{F}^{\mathfrak{c}}_{m-2}\mathrm{Conv}(\mathfrak{oc}^{\vee},\mathsf{KGra}), \tag{8.79}$$

and the fact that $\psi_{m-1,k'+1}$ satisfies Properties A.1, A.2, it is not hard to show that

$$[\psi^{(2)},\alpha] \in \mathcal{F}^{\mathfrak{c}}_{m-1}\mathrm{Conv}(\mathfrak{oc}^{\vee},\mathsf{KGra})\,. \tag{8.80}$$

Hence we do have inclusion (8.60).

Next, using (8.67), (8.80), and the inequality $m \ge 2$, we deduce that

$$\xi^{(2)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k}) = \xi^{(1)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k}) + [\psi^{(2)},\alpha](\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k}) \tag{8.81}$$

Since equation (8.57) holds for all $k < k'$, it is obvious that (8.61) holds for all $k < k'$.

For $\mathbf{t}^{\mathfrak{o}}_{m,k'}$ we have

$$\xi^{(2)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k'}) = \xi^{(1)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k'}) + [\psi^{(2)},\alpha](\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k'}) =$$

$$\xi^{(1)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k'}) + k'\sum_{i=1}^{m}(\tau_{m,i},\mathrm{id})\big(\psi_{m-1,k'+1}\circ_{\mathfrak{o},1}\Gamma^{\mathrm{br}}_0\big) =$$

$$\xi^{(1)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k'}) + \mathfrak{d}(\psi_{m-1,k'+1})\,.$$

Thus

$$\xi^{(2)}(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k'}) = 0 \tag{8.82}$$

due to (8.76).

We conclude that (8.61) holds for all $k \le k'$. Step 2 is complete.

Upon completing Steps 1 and 2, we proved that there exists a degree $-1$ vector (8.52) such that conditions (8.53) and (8.54) hold.

Thus the desired vector $\xi'$ in the claim is the following limit

$$\xi' = \lim_{K \to \infty} \mathrm{CH}\Big( \dots \mathrm{CH}\big( \mathrm{CH}(\xi, [\psi_{k'}, \alpha]) \,,\, [\psi_{k'+1}, \alpha] \big), \dots, [\psi_K, \alpha] \Big) \tag{8.83}$$

which exists since $\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$ is complete with respect to the arity filtration.

Claim 8.5 is proved.

$\square$

Thus, if $m < n-1$ then, applying Claim 8.5 enough times we would get to the situation when $m = n-1$ and we would arrive at a contradiction found in Section 8.3. This contradiction implies that the action of $\exp\big(H^0(\mathsf{dfGC})\big)$ on homotopy classes of stable formality quasi-isomorphisms is indeed faithful.

Theorem 6.2 is proved.

# 9 A modification: loopless version of stable formality quasi-isomorphisms

We could have discarded loops from our consideration from the very beginning. This way we would get operads $\mathsf{dGra}^{\not\circlearrowleft}$, $\mathsf{KGra}^{\not\circlearrowleft}$, as well as the loopless version of the full graph complex

$$\mathsf{dfGC}^{\not\circlearrowleft} = \mathrm{Conv}(\Lambda^2\mathsf{coCom}, \mathsf{dGra}^{\not\circlearrowleft}) \,. \tag{9.1}$$

Proceeding in this spirit, we say that a stable formality quasi-isomorphism $F$ is *loopless* if the morphism $F$ lands in $\mathsf{KGra}^{\not\circlearrowleft}$. It is clear how to define the notion of homotopy equivalence between two loopless stable formality quasi-isomorphisms. In fact, in the loopless setting, condition (5.14) holds automatically for all degree zero vectors

$$\xi \in \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}) \,.$$

Thus one should only impose on $\xi$ condition (5.13).

Following the same line of arguments, it is easy to prove a version of Theorem 6.2 in this loopless setting. Namely,

**Theorem 9.1** *The pro-unipotent group* $\exp(H^0(\mathsf{dfGC}^{\not\circlearrowleft}))$ *acts simply transitively on the set of homotopy classes of loopless stable formality quasi-isomorphisms.* $\square$

# A A cochain complex that is closely connected with the Hochschild complex of a cofree cocommutative coalgebra

In this appendix we compute the cohomology of the cochain complex

$$\mathsf{KGra}^{\mathrm{Hoch}}_{\mathrm{inv}} = \mathbf{s}^{2n-2} \bigoplus_{k \ge 0} \mathbf{s}^k \big(\mathsf{KGra}(n,k)^{\mathfrak{o}}\big)^{S_n} \tag{A.1}$$

with the differential $\partial^{\mathrm{Hoch}}$ given by the formula

$$\partial^{\mathrm{Hoch}}(\gamma) = \Gamma_{\circ\circ} \circ_{2,\mathfrak{o}} \gamma - \gamma \circ_{1,\mathfrak{o}} \Gamma_{\circ\circ} + \gamma \circ_{2,\mathfrak{o}} \Gamma_{\circ\circ} - \dots \tag{A.2}$$

$$+(-1)^k \gamma \circ_{k,\mathfrak{o}} \Gamma_{\circ\circ} (-1)^{k+1} \Gamma_{\circ\circ} \circ_{1,\mathfrak{o}} \gamma \,,$$

$$\gamma \in \mathbf{s}^{2n-2+k} \big(\mathsf{KGra}(n,k)^{\mathfrak{o}}\big)^{S_n} \,.$$

For this purpose we consider a slightly simpler cochain complex

$$\mathsf{KGra}^{\mathrm{Hoch}} = \mathbf{s}^{2n-2} \bigoplus_{k \ge 0} \mathbf{s}^k \mathsf{KGra}(n,k)^{\mathfrak{o}} \tag{A.3}$$

with the differential $\partial^{\mathrm{Hoch}}$ defined by the same formula (A.2).

The cochain complex (A.3) is equipped with the obvious action of the group $S_n$ and (A.1) is nothing but the complex of $S_n$-invariants.

**Example A.1** An example of computation of $\partial^{\mathrm{Hoch}}(\Gamma)$ for a graph $\Gamma \in \mathrm{dgra}_{3,1}$ is shown on figure A.1. Let us say that we chose this order $(1_{\mathfrak{c}}, 3_{\mathfrak{c}}) < (1_{\mathfrak{c}}, 1_{\mathfrak{o}}) < (2_{\mathfrak{c}}, 1_{\mathfrak{o}}) < (3_{\mathfrak{c}}, 1_{\mathfrak{o}})$ on

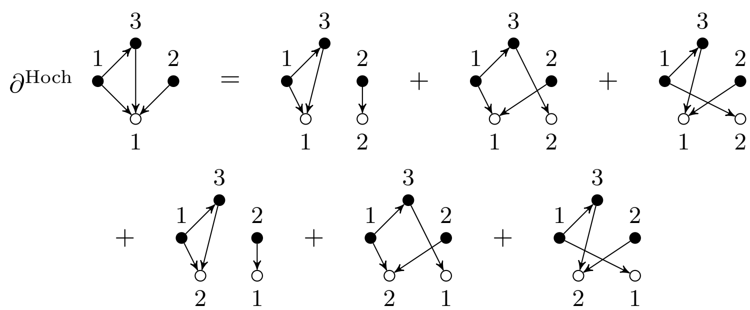


Fig. A.1: Computing $\partial^{\mathrm{Hoch}}$

the set of edges of $\Gamma$. The orders on the sets of edges of graphs in the right hand side are inherited from the total order on the edges of $\Gamma$ in the obvious way. For example, the first graph in the sum on the right hand side has its edges ordered this way: $(1_{\mathfrak{c}}, 3_{\mathfrak{c}}) < (1_{\mathfrak{c}}, 1_{\mathfrak{o}}) < (2_{\mathfrak{c}}, 2_{\mathfrak{o}}) < (3_{\mathfrak{c}}, 1_{\mathfrak{o}})$.

Before computing the cohomology of (A.3) let us make a couple of remarks about vectors

$$c \in \mathbf{s}^{2n-2+k} \mathsf{KGra}(n,k)^{\mathfrak{o}} \quad \text{or} \quad c \in \mathbf{s}^{2n-2+k} \big(\mathsf{KGra}(n,k)^{\mathfrak{o}}\big)^{S_n} \tag{A.4}$$

satisfying these two properties:

**Property A.1** All white vertices in each graph of the linear combination $c$ have valency one.

**Property A.2** For every $\sigma \in S_k$ we have

$$(\mathrm{id}, \sigma)\,(c) = (-1)^{|\sigma|} c \,. \tag{A.5}$$

For example, the "brooms" $\Gamma^{\mathrm{br}}_k$ depicted on figure 5.1 obviously satisfy these properties.

**Remark A.2** It is easy to see that every vector (A.4) satisfying Properties A.1 and A.2 is closed with respect to $\partial^{\mathrm{Hoch}}$. Furthermore, it is not hard to see that a cocycle $c$ satisfying Properties A.1 and A.2 is trivial if and only if $c = 0$.

### A.1 The Hochschild complex of a cofree cocommutative coalgebra

To compute the cohomology of (A.3) we consider the cofree cocommutative $\mathbb{K}$-coalgebra $\mathscr{C}_r$ with counit co-generated by degree 0 elements $h_1, h_2, \dots, h_r$.

To the coalgebra $\mathscr{C}_r$ we assign the following cochain complex

$$\mathrm{Hoch}(\mathscr{C}_r) = \bigoplus_{k \ge 0} \mathbf{s}^k \big(\mathscr{C}_r\big)^{\otimes k} \tag{A.6}$$

with the differential

$$\partial^{\mathscr{C}} : \big(\mathscr{C}_r\big)^{\otimes k} \to \big(\mathscr{C}_r\big)^{\otimes (k+1)}$$

given by the formula

$$\partial^{\mathscr{C}}(X) = 1 \otimes X + \sum_{i=1}^{k} (-1)^i (\mathrm{id}, \dots, \mathrm{id}, \underbrace{\Delta}_{i\text{-th spot}}, \mathrm{id}, \dots, \mathrm{id})(X) + (-1)^{k+1} X \otimes 1\,, \tag{A.7}$$

where $\Delta$ denotes the comultiplication on $\mathscr{C}_r$.

The complex $\mathrm{Hoch}(\mathscr{C}_r)$ obviously splits into the direct sum of sub-complexes

$$\mathrm{Hoch}(\mathscr{C}_r) = \bigoplus_{m \ge 0} \mathrm{Hoch}(\mathscr{C}_r)_m\,, \tag{A.8}$$

where $\mathrm{Hoch}(\mathscr{C}_r)_m$ is spanned by tensor monomials with the total degree in co-generators being $m$.

In [24, Section 4.6.1.1] it was proved that

**Claim A.1 (Section 4.6.1.1, [24])** *If $X$ is a cocycle in*

$$\mathbf{s}^k \big(\mathscr{C}_r\big)^{\otimes k} \ \cap\ \mathrm{Hoch}(\mathscr{C}_r)_m$$

*and $m \neq k$ then $X$ is $\partial^{\mathscr{C}}$-exact. Furthermore, if $X$ is a cocycle in*

$$\mathbf{s}^k \big(\mathscr{C}_r\big)^{\otimes k} \ \cap\ \mathrm{Hoch}(\mathscr{C}_r)_m$$

*and $m = k$ then there exists*

$$\widetilde{X} \in \mathbf{s}^{k-1} \big(\mathscr{C}_r\big)^{\otimes (k-1)} \ \cap\ \mathrm{Hoch}(\mathscr{C}_r)_m$$

*such that*

$$X - \partial^{\mathscr{C}}(\widetilde{X}) = \sum_{i_1 i_2 \dots i_k} \lambda^{i_1 i_2 \dots i_k} (h_{i_1}, h_{i_2}, \dots, h_{i_k})\,,$$

*where $\lambda^{i_1 i_2 \dots i_k} \in \mathbb{K}$ and*

$$\lambda^{\dots i_p i_{p+1} \dots} = -\lambda^{\dots i_{p+1} i_p \dots}\,.$$

*Finally a cocycle of the form*

$$\sum_{i_1 i_2 \dots i_k} \lambda^{i_1 i_2 \dots i_k} (h_{i_1}, h_{i_2}, \dots, h_{i_k})\,, \qquad \lambda^{\dots i_p i_{p+1} \dots} = -\lambda^{\dots i_{p+1} i_p \dots} \in \mathbb{K}$$

*is exact if and only if all coefficients $\lambda^{i_1 i_2 \dots i_k} = 0$.* $\square$

For our purposes we will need the following subcomplex of $\mathrm{Hoch}(\mathscr{C}_r)$:

$$\mathrm{Hoch}'(\mathscr{C}_r) = \Big\{ X \in \mathrm{Hoch}(\mathscr{C}_r)_r \ \Big| \ \text{each co-generator } h_i \text{ appears} \tag{A.9}$$

$$\text{in the tensor monomial } X \text{ exactly once} \Big\} .$$

Using Claim A.1 about cocycles in $\mathrm{Hoch}(\mathscr{C}_r)$ it is easy to deduce an analogous statement for the cochain complex $\mathrm{Hoch}'(\mathscr{C}_r)$:

**Claim A.2** *If $X$ is a cocycle in*

$$\mathbf{s}^k \big(\mathscr{C}_r\big)^{\otimes k} \ \cap \ \mathrm{Hoch}'(\mathscr{C}_r)$$

*and $k \neq r$ then $X$ is $\partial^{\mathscr{C}}$-exact. Furthermore, if $X$ is a cocycle in*

$$\mathbf{s}^k \big(\mathscr{C}_r\big)^{\otimes k} \ \cap \ \mathrm{Hoch}'(\mathscr{C}_r)$$

*and $k = r$ then there exists*

$$\widetilde{X} \in \mathbf{s}^{k-1} \big(\mathscr{C}_r\big)^{\otimes (k-1)} \ \cap \ \mathrm{Hoch}'(\mathscr{C}_r)$$

*such that*

$$X - \partial^{\mathscr{C}}(\widetilde{X}) = \sum_{\sigma \in S_r} (-1)^{|\sigma|} \lambda (h_{\sigma(1)}, h_{\sigma(2)}, \dots, h_{\sigma(r)}) \,,$$

*for $\lambda \in \mathbb{K}$. Finally, the cocycle*

$$\sum_{\sigma \in S_r} (-1)^{|\sigma|} (h_{\sigma(1)}, h_{\sigma(2)}, \dots, h_{\sigma(r)})$$

*is non-trivial.* $\square$

## A.2 Computing cohomology of $\mathsf{KGra}^{\mathrm{Hoch}}$ and $\mathsf{KGra}^{\mathrm{Hoch}}_{\mathrm{inv}}$

Let us now return to the cochain complex $\mathsf{KGra}^{\mathrm{Hoch}}$ (A.3).

It is clear that $\mathsf{KGra}^{\mathrm{Hoch}}$ splits into the direct sum of sub-complexes

$$\mathsf{KGra}^{\mathrm{Hoch}} = \bigoplus_r \mathsf{KGra}^{\mathrm{Hoch}}_r \tag{A.10}$$

where $\mathsf{KGra}^{\mathrm{Hoch}}_r$ is spanned by graphs with exactly $r$ edges terminating at white vertices.

To compute the cohomology of $\mathsf{KGra}^{\mathrm{Hoch}}_r$ we introduce an auxiliary subspace:

$$\mathsf{KGra}'(n,r) \subset \mathsf{KGra}(n,r) \tag{A.11}$$

which consists of linear combinations of graphs in $\mathrm{dgra}_{n,r}$ with all white vertices (if any) having valency $1$.

Let us now suppose that we are given a tensor monomial with $k$ factors

$$X = h_{i_{11}} h_{i_{12}} \dots h_{i_{1r_1}} \otimes h_{i_{21}} h_{i_{22}} \dots h_{i_{2r_2}} \otimes \dots \otimes h_{i_{k1}} h_{i_{k2}} \dots h_{i_{kr_k}} \in \mathrm{Hoch}'(\mathscr{C}_r) \tag{A.12}$$

and a graph $\Gamma' \in \mathrm{dgra}_{n,r}$ with all white vertices having valency $1$. To the pair $(X, \Gamma')$ we assign a graph $\Gamma \in \mathrm{dgra}_{n,k}$ following these steps:

- First, for each $i \in \{1, 2, \dots, r\}$ we find the number of the tensor factor in (A.12) which contains the co-generator[14] $h_i$. We denote this number by $d_i$.
- Second, we erase white vertices of $\Gamma'$ and attach the resulting free edges to new $k$ white vertices with labels $1, 2, \dots, k$ following this rule: the edged which previously terminated at the white vertex with label $i$ should now terminate at the white vertex with label $d_i$.
- Finally, in the resulting graph $\Gamma$, we keep the same total order on the set of edges as for $\Gamma'$.

**Example A.3** To a graph $\Gamma'$ depicted on figure A.2 and the monomial

$$(h_1 h_2, 1, h_3, 1) \in \mathrm{Hoch}'(\mathscr{C}_3)$$

we should assign the graph $\Gamma$ shown on figure A.3. The total order on the set of edges of $\Gamma$ is inherited from the total order on the set of edges of $\Gamma'$.

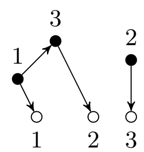



Fig. A.2: A graph $\Gamma' \in \mathrm{dgra}_{3,3}$

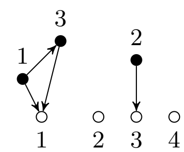



Fig. A.3: The graph $\Gamma \in \mathrm{dgra}_{3,4}$

The described procedure gives us an obvious map

$$\Upsilon' : \mathbf{s}^{2n-2}\mathsf{KGra}'(n, r) \otimes \mathrm{Hoch}'(\mathscr{C}_r) \to \mathsf{KGra}^{\mathrm{Hoch}}_r . \tag{A.13}$$

The group $S_r$ acts in the obvious way on the source of the map (A.13) by simultaneously rearranging the labels on white vertices and co-generators of $\mathscr{C}_r$. It is easy to see that $\Upsilon'$ (A.13) descends to an isomorphism

$$\Upsilon : \big(\mathbf{s}^{2n-2}\mathsf{KGra}'(n, r) \otimes \mathrm{Hoch}'(\mathscr{C}_r)\big)_{S_r} \to \mathsf{KGra}^{\mathrm{Hoch}}_r . \tag{A.14}$$

from the space

$$\big(\mathbf{s}^{2n-2}\mathsf{KGra}'(n, r) \otimes \mathrm{Hoch}'(\mathscr{C}_r)\big)_{S_r}$$

of $S_r$-coinvariants to the complex in question $\mathsf{KGra}^{\mathrm{Hoch}}_r$. It is not hard to see that the map (A.14) is compatible with the differential $\partial^{\mathrm{Hoch}}$ on $\mathsf{KGra}^{\mathrm{Hoch}}_r$ and the differential on the source coming from $\partial^{\mathscr{C}}$ on $\mathrm{Hoch}'(\mathscr{C}_r)$.

Thus, using Claim A.2, it is not hard to prove the following statement about cohomology of $\mathsf{KGra}^{\mathrm{Hoch}}$ (A.3).

**Proposition A.1** *For every cocycle*

$$\gamma \in \mathbf{s}^{2n-2+k}\mathsf{KGra}(n, k)^{\mathfrak{o}}$$

*there exists a vector*

$$\gamma_1 \in \mathbf{s}^{2n-2+k-1}\mathsf{KGra}(n, k-1)^{\mathfrak{o}}$$

[14] Recall that each co-generator $h_i$ enters the monomial (A.12) exactly once.

*such that the difference*

$$c = \gamma - \partial^{\mathrm{Hoch}}(\gamma_1)$$

*satisfies Properties A.1 and A.2. A cocycle c in* (A.3) *satisfying Properties A.1 and A.2 is trivial if and only if* $c = 0$ . $\square$

To deduce an analogous statement for the cochain complex $\mathsf{KGra}^{\mathrm{Hoch}}_{\mathrm{inv}}$ (A.1) we need to use the averaging operator

$$\frac{1}{n!}\sum_{\sigma\in S_n}\sigma\,.$$

More precisely, Proposition A.1 implies that

**Corollary A.1** *For every cocycle*

$$\gamma \in \mathbf{s}^{2n-2+k}\Big(\mathsf{KGra}(n,k)^{\mathfrak{o}}\Big)^{S_n}$$

*there exists a vector*

$$\gamma_1 \in \mathbf{s}^{2n-2+k-1}\Big(\mathsf{KGra}(n,k-1)^{\mathfrak{o}}\Big)^{S_n}$$

*such that the difference*

$$c = \gamma - \partial^{\mathrm{Hoch}}(\gamma_1)$$

*satisfies Properties A.1 and A.2. A cocycle c in the complex* (A.1) *satisfying Properties A.1 and A.2 is trivial if and only if* $c = 0$ . $\square$

It is clear that for every vector

$$\gamma \in \mathbf{s}^{2n-2}\Big(\mathsf{KGra}(n,0)^{\mathfrak{o}}\Big)^{S_n}$$

$$\partial^{\mathrm{Hoch}}(\gamma) = 0\,. \tag{A.15}$$

Due to this observation Corollary A.1 implies the following statement.

**Corollary A.2** *A vector*

$$\gamma \in \mathbf{s}^{2n-1}\Big(\mathsf{KGra}(n,1)^{\mathfrak{o}}\Big)^{S_n} \tag{A.16}$$

*is a cocycle in* (A.1) *if and only if the white vertex in each graph in the linear combination* $\gamma$ *has valency* $1$ . *Furthermore, a cocycle* $\gamma$ *in* $\mathbf{s}^{2n-1}\Big(\mathsf{KGra}(n,1)^{\mathfrak{o}}\Big)^{S_n}$ *is trivial if and only if* $\gamma = 0$ . $\square$

## B The complex of "hedgehogs"

This appendix is devoted to an auxiliary cochain complex which is assembled from graphs $\Gamma \in \mathrm{dgra}_{m,k}$ satisfying the additional property: *each white vertex of* $\Gamma$ *has valency* $1$ . Since such graphs look like hedgehogs we call this cochain complex the complex of "hedgehogs".

This cochain complex and especially Corollary B.1 (proved below) are used in the proof of Claim 7.5.

We start by introducing the following graded vector space

$$\mathsf{Hg} = \Big\{ \gamma \in \bigoplus_{m,k} \mathbf{s}^{2m-2+k} \big(\mathsf{KGra}(m,k)^{\mathfrak{o}}\big)^{S_m} \quad \Big| \quad \gamma \text{ obeys Properties A.1, A.2} \Big\} \tag{B.1}$$

and the families of cycles $\tau_{m,i} \in S_m$, and $\sigma_{k,i}, \varsigma_{k,i} \in S_k$

$$\tau_{m,i} = \begin{pmatrix} 1 & \dots & i-1 & i & \dots & m-1 & m \\ 1 & \dots & i-1 & i+1 & \dots & m & i \end{pmatrix}, \tag{B.2}$$

$$\sigma_{k,i} = \begin{pmatrix} 1 & 2 & \dots & i & i+1 & \dots & k \\ i & 1 & \dots & i-1 & i+1 & \dots & k \end{pmatrix}, \tag{B.3}$$

and

$$\varsigma_{k,i} = \begin{pmatrix} 1 & 2 & \dots & i-1 & i & i+1 & \dots & k \\ 2 & 3 & \dots & i & 1 & i+1 & \dots & k \end{pmatrix}. \tag{B.4}$$

Next, we denote by $\mathfrak{d}$ the following degree 1 operation on $\mathsf{Hg}$

$$\mathfrak{d}(\gamma) = k \sum_{i=1}^{m+1} (\tau_{m+1,i}, \mathrm{id}) \big( \gamma \circ_{1,\mathfrak{o}} \Gamma_0^{\mathrm{br}} \big), \qquad \gamma \in \mathbf{s}^{2m-2+k} \big(\mathsf{KGra}(m,k)^{\mathfrak{o}}\big)^{S_m}. \tag{B.5}$$

Notice that, since the graph $\Gamma_0^{\mathrm{br}}$ consists of a single black vertex and has no edges, the insertion $\circ_{1,\mathfrak{o}}$ of $\Gamma_0^{\mathrm{br}}$ replaces the white vertex with label 1 by a black vertex with label $m+1$ and shifts the labels on the remaining white vertices down by 1.

Using the fact that each linear combination $\gamma \in \mathsf{Hg}$ is anti-symmetric with respect to permutations of labels on white vertices, it is not hard to deduce that

$$\mathfrak{d}^2 = 0. \tag{B.6}$$

Thus $(\mathsf{Hg}, \mathfrak{d})$ is a cochain complex. We call this cochain complex the complex of "hedgehogs".

For our purposes, we need a degree $-1$ operation

$$\mathfrak{d}^* : \mathsf{Hg} \to \mathsf{Hg} \tag{B.7}$$

which we will now define. Let $\gamma$ be a vector in $\mathbf{s}^{2m-2+k}\big(\mathsf{KGra}(m,k)^{\mathfrak{o}}\big)^{S_m}$ satisfying Properties A.1, A.2. To compute $\mathfrak{d}^*(\gamma)$ we follow these steps:

- First, we omit in $\gamma$ all graphs for which the black vertex with label 1 is not a pike. We denote the resulting linear combination in $\mathbf{s}^{2m-2+k}\mathsf{KGra}(m,k)^{\mathfrak{o}}$ by $\gamma'$.
- Second, we replace the black vertex with label 1 in each graph of $\gamma'$ by a white vertex and shift all labels on black vertices down by 1. We assign label 1 to this additional white vertex and shift the labels of the remaining white vertices up by 1. We denote the resulting linear combination in $\mathbf{s}^{2(m-1)-2+k+1}\mathsf{KGra}(m-1,k+1)^{\mathfrak{o}}$ by $\gamma''$.
- Finally, we set

$$\mathfrak{d}^*(\gamma) = \sum_{i=1}^{k+1} \frac{(-1)^{i-1}}{k+1} (\mathrm{id}, \sigma_{k+1,i})(\gamma''). \tag{B.8}$$

It is easy to see that the linear combination $\mathfrak{d}^*(\gamma)$ is $S_{m-1}$-invariant and satisfies Properties A.1, A.2.

**Remark B.1** Notice that

$$\mathfrak{d}^*(\gamma) = 0\,. \tag{B.9}$$

if each graph in the linear combination $\gamma$ does not have pikes.

**Example B.2** Let us denote by $\Gamma_k$ the graph depicted on figure B.1 and let

$$\gamma = \Gamma_k + (\sigma_{12}, \mathrm{id})\big(\Gamma_k\big)\,, \tag{B.10}$$

where $\sigma_{12}$ is the transposition in $S_2$.

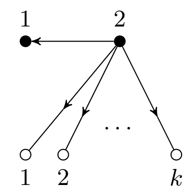



Fig. B.1: Edges are ordered in this way $(2_{\mathfrak{c}}, 1_{\mathfrak{c}}) < (2_{\mathfrak{c}}, 1_{\mathfrak{o}}) < (2_{\mathfrak{c}}, 2_{\mathfrak{o}}) < \dots < (2_{\mathfrak{c}}, k_{\mathfrak{o}})$

It is obvious that $\gamma$ is a vector in $\mathbf{s}^{k+2}\big(\mathsf{KGra}(2,k)^{\mathfrak{o}}\big)^{S_m}$ satisfying Properties A.1, A.2. Following the steps outlined above, we get

$$\gamma' = \Gamma_k \qquad \text{and} \qquad \gamma'' = \Gamma^{\mathrm{br}}_{k+1}\,,$$

where $\Gamma^{\mathrm{br}}_k$ is the family of "brooms" shown on figure 5.1 . Since $\Gamma^{\mathrm{br}}_{k+1}$ is already antisymmetric with respect to permutation of labels on white vertices,

$$\mathfrak{d}^*(\gamma) = \sum_{i=1}^{k+1} \frac{(-1)^{i-1}}{k+1} (\mathrm{id}, \sigma_{k+1,i})(\Gamma^{\mathrm{br}}_{k+1}) = \Gamma^{\mathrm{br}}_{k+1}\,.$$

We need the following lemma.

**Lemma B.3** *For every vector*

$$\gamma \in \mathbf{s}^{2m-2+k}\big(\mathsf{KGra}(m,k)^{\mathfrak{o}}\big)^{S_m}$$

*satisfying Properties A.1, A.2 we have*

$$\mathfrak{d}\mathfrak{d}^*(\gamma) + \mathfrak{d}^*\mathfrak{d}(\gamma) = k\gamma + \sum_{r \ge 1} r\gamma_r\,, \tag{B.11}$$

*where $\gamma_r$ is the linear combination in* $\mathsf{Hg}$ *which is obtained from $\gamma$ by retaining the graphs with exactly $r$ pikes.*

Proof. Let us observe that the space

$$\mathbf{s}^{2m-2+k}\big(\mathsf{KGra}(m,k)^{\mathfrak{o}}\big)^{S_m}$$

is spanned by vectors of the form

$$\sum_{\tau \in S_m,\, \sigma \in S_k} (-1)^{|\sigma|} (\tau, \sigma)\big(\Gamma\big) \tag{B.12}$$

where $\Gamma$ is a graph in $\mathrm{dgra}_{m,k}$ with all white vertices having valency $1$.

Thus we may assume, without loss of generality that

$$\gamma = \sum_{\tau \in S_m,\, \sigma \in S_k} (-1)^{|\sigma|} (\tau, \sigma)\big(\Gamma\big) \tag{B.13}$$

for a graph $\Gamma \in \mathrm{dgra}_{m,k}$ with all white vertices having valency $1$.

Using the cycles $\varsigma_{k,i}$ (B.4) we rewrite (B.13) as follows:

$$\gamma = \sum_{\tau \in S_m} \sum_{\sigma' \in S_{\{2,3,\dots,k\}}} (-1)^{|\sigma'|} (\tau, \sigma') \left( \sum_{i=1}^{k} (-1)^{i-1} (\mathrm{id}, \varsigma_{k,i})(\Gamma) \right) , \tag{B.14}$$

where $S_{\{2,3,\dots,k\}}$ denotes the permutation group of the set $\{2, 3, \dots, k\}$.

Next, using (B.14) together with the obvious identity

$$\big((\mathrm{id}, \varsigma_{k,i})(\Gamma)\big) \circ_{1,\mathfrak{o}} \Gamma^{\mathrm{br}}_0 = \Gamma \circ_{i,\mathfrak{o}} \Gamma^{\mathrm{br}}_0$$

we deduce that

$$\mathfrak{d}(\gamma) = k \sum_{j=1}^{m+1} (\tau_{m+1,j}, \mathrm{id}) \left( \sum_{\tau \in S_m,\, \sigma' \in S_{k-1}} (-1)^{|\sigma'|} (\tau, \sigma') \left( \sum_{i=1}^{k} (-1)^{i-1} \Gamma \circ_{i,\mathfrak{o}} \Gamma^{\mathrm{br}}_0 \right) \right) =$$

$$k \sum_{\tau \in S_{m+1},\, \sigma' \in S_{k-1}} (-1)^{|\sigma'|} (\tau, \sigma') \left( \sum_{i=1}^{k} (-1)^{i-1} \Gamma \circ_{i,\mathfrak{o}} \Gamma^{\mathrm{br}}_0 \right) . \tag{B.15}$$

Let us, first, consider the case when the graph $\Gamma$ does not have pikes. In this case, due to Remark B.1, we have

$$\mathfrak{d}^*(\gamma) = 0 .$$

Furthermore, using (B.15), we get

$$\mathfrak{d}^*\mathfrak{d}(\gamma) =$$

$$\sum_{j=1}^{k} (-1)^{j-1} (\mathrm{id}, \sigma_{k,j}) \left( \sum_{\tau \in S_m} \sum_{\sigma' \in S_{\{2,3,\dots,k\}}} (-1)^{|\sigma'|} (\tau, \sigma') \left( \sum_{i=1}^{k} (-1)^{i-1} (\mathrm{id}, \varsigma_{k,i})(\Gamma) \right) \right) , \tag{B.16}$$

where $S_{\{2,3,\dots,k\}}$ denotes the permutation group of the set $\{2, 3, \dots, k\}$, and $\varsigma_{k,i}$ is the family of cycles defined in (B.4).

It is not hard to see that

$$\sum_{\tau \in S_m} \sum_{\sigma' \in S_{\{2,3,\dots,k\}}} (-1)^{|\sigma'|} (\tau, \sigma') \left( \sum_{i=1}^{k} (-1)^{i-1} (\mathrm{id}, \varsigma_{k,i})(\Gamma) \right)$$

$$= \sum_{\tau \in S_m,\, \sigma \in S_k} (-1)^{|\sigma|} (\tau, \sigma)\big(\Gamma\big) = \gamma . \tag{B.17}$$

On the other hand,

$$(\mathrm{id}, \sigma_{k,j})(\gamma) = (-1)^{j-1} \gamma$$

because $\gamma$ is antisymmetric with respect to permutations of labels on white vertices. Hence,

$$\sum_{j=1}^{k}(-1)^{j-1}(\mathrm{id},\sigma_{k,j})(\gamma)=k\gamma\,. \tag{B.18}$$

Therefore combining (B.16) with (B.17) and (B.18) we get

$$\mathfrak{d}^*\mathfrak{d}(\gamma)=k\gamma\,. \tag{B.19}$$

Thus, if each graph in a linear combination $\gamma$ does not have pikes then equation (B.11) holds.

Let us now turn to the case when $\Gamma$ has exactly $r\geq 1$ pikes.

Without loss of generality, we may assume that the pikes of $\Gamma$ are labeled by $1,2,\dots,r$.

Let us recall that the vector $\gamma'$ is obtained from $\gamma$ by discarding all graphs for which the black vertex with label 1 is not a pike. In our case, the vector $\gamma'$ can be written as follows:

$$\gamma'=\sum_{\sigma\in S_k}\sum_{\tau'\in S_{\{2,3,\dots,m\}}}(-1)^{|\sigma|}(\tau',\sigma)\left(\sum_{p=1}^{r}(\varsigma_{m,p},\mathrm{id})(\Gamma)\right)\,, \tag{B.20}$$

where $S_{\{2,3,\dots,m\}}$ denotes the permutation group of the set $\{2,3,\dots,m\}$, and $\varsigma_{m,p}$ is the family of cycles in $S_m$ defined in (B.4).

Using (B.20) we get

$$\mathfrak{d}^*(\gamma)=\sum_{i=1}^{k+1}\frac{(-1)^{i-1}}{k+1}(\mathrm{id},\sigma_{k+1,i})(\gamma'') \tag{B.21}$$

with

$$\gamma''=\sum_{\sigma\in S_{\{2,3,\dots,k+1\}}}\sum_{\tau'\in S_{m-1}}(-1)^{|\sigma|}(\tau',\sigma)\left(\sum_{p=1}^{r}R_\circ\big((\varsigma_{m,p},\mathrm{id})(\Gamma)\big)\right)\,, \tag{B.22}$$

where $S_{\{2,3,\dots,k+1\}}$ is the group of permutations of the set $\{2,3,\dots,k+1\}$, and $R_\circ$ is the operation which replaces the pike with label 1 by a white vertex with label 1, shifts labels on the remaining white vertices up by 1 and shifts labels on black vertices down by 1.

For the vector $\mathfrak{d}\mathfrak{d}^*(\gamma)$ we get

$$\mathfrak{d}\mathfrak{d}^*(\gamma)=\sum_{j=1}^{m}(\tau_{m,j}\,,\,\mathrm{id})\left(\sum_{\sigma\in S_k}\sum_{\tau'\in S_{m-1}}(-1)^{|\sigma|}(\tau',\sigma)\left(\sum_{p=1}^{r}(\varsigma_{m,p},\mathrm{id})(\Gamma)\right)\right)+ \tag{B.23}$$

$$+\sum_{j=1}^{m}(\tau_{m,j},\mathrm{id})\left(\sum_{i=2}^{k+1}(-1)^{i-1}\big((\mathrm{id},\sigma_{k+1,i})(\gamma'')\big)\circ_{1,\mathfrak{o}}\Gamma_0^{\mathrm{br}}\right)\,,$$

where the first sum comes from the first term in the sum (B.21) and the second sum comes from the remaining terms in (B.21).

The first sum in (B.23) can be simplified as follows.

$$\sum_{j=1}^{m}(\tau_{m,j}\,,\,\mathrm{id})\left(\sum_{\sigma\in S_k}\sum_{\tau'\in S_{m-1}}(-1)^{|\sigma|}(\tau',\sigma)\left(\sum_{p=1}^{r}(\varsigma_{m,p},\mathrm{id})(\Gamma)\right)\right)=$$

$$\sum_{\sigma\in S_k}\sum_{\tau\in S_m}(-1)^{|\sigma|}(\tau,\sigma)\left(\sum_{p=1}^{r}(\varsigma_{m,p},\mathrm{id})(\Gamma)\right)=$$
$$r\sum_{\sigma\in S_k}\sum_{\tau\in S_m}(-1)^{|\sigma|}(\tau,\sigma)(\Gamma)=r\,\gamma\,.$$

In other words,

$$\sum_{j=1}^{m}(\tau_{m,j}\,,\,\mathrm{id})\left(\sum_{\sigma\in S_k}\sum_{\tau'\in S_{m-1}}(-1)^{|\sigma|}(\tau',\sigma)\left(\sum_{p=1}^{r}(\varsigma_{m,p},\mathrm{id})(\Gamma)\right)\right)=r\,\gamma \tag{B.24}$$

To simplify the second sum in (B.23) we notice that the subsets of $S_{k+1}$

$$\{\sigma_{k+1,i}\circ\sigma \mid \sigma\in S_{\{2,3,\dots,k+1\}},\ 2\le i\le k+1\}$$

and

$$\{\sigma\circ\varsigma_{k+1,i} \mid \sigma\in S_{\{2,3,\dots,k+1\}},\ 2\le i\le k+1\}$$

coincide.

Hence,

$$\sum_{i=2}^{k+1}\frac{(-1)^{i-1}}{k+1}(\mathrm{id},\sigma_{k+1,i})(\gamma'')=$$
$$\frac{1}{k+1}\sum_{\sigma\in S_{\{2,3,\dots,k+1\}}}\sum_{\tau'\in S_{m-1}}(-1)^{|\sigma|}(\tau',\sigma)\left(\sum_{p=1}^{r}\sum_{i=2}^{k+1}(-1)^{i-1}(\mathrm{id},\varsigma_{k+1,i})R_\circ\big((\varsigma_{m,p},\mathrm{id})(\Gamma)\big)\right)\,. \tag{B.25}$$

Next, we introduce operations $\{\mathrm{Cg}_p^i\}_{1\le p\le r, 1\le i\le k}$ whose input is our graph $\Gamma$ and whose outputs are graph in $\mathrm{dgra}_{m,k}$ with the same properties, i.e. each white vertex of $\mathrm{Cg}_p^i(\Gamma)$ has valency 1 and $\mathrm{Cg}_p^i(\Gamma)$ has exactly $r$ pikes. This operation is illustrated on figure B.2. More

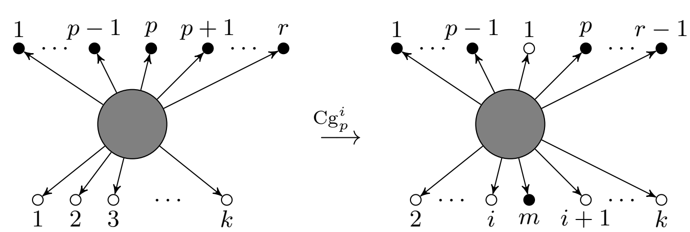



Fig. B.2: The operation $\Gamma\mapsto\mathrm{Cg}_p^i(\Gamma)$. Gray regions denote subgraphs formed by black vertices which are not pikes

precisely, $\mathrm{Cg}_p^i(\Gamma)$ is obtained from $\Gamma$ via these steps:

- first, we replace the black vertex with label $p$ by a white vertex and replace the white vertex with label $i$ by a black vertex;
- second, we shift the labels on the black vertices which are $> p$ down by 1;
- third, we shift the labels on the white vertices which are $< i$ up by 1;

- finally, we assign label 1 to the new white vertex and we assign label $m$ to the new black vertex.

Using equation (B.25) and the graphs $\mathrm{Cg}^i_p(\Gamma)$ we present the second sum in (B.23) in the following way.

$$\sum_{j=1}^{m}(\tau_{m,j},\mathrm{id})\left(\sum_{i=2}^{k+1}(-1)^{i-1}\big((\mathrm{id},\sigma_{k+1,i})(\gamma'')\big)\circ_{1,\mathfrak{o}}\Gamma_0^{\mathrm{br}}\right)= \tag{B.26}$$

$$\sum_{j=1}^{m}(\tau_{m,j},\mathrm{id})\sum_{\tau'\in S_{m-1}}\sum_{\sigma\in S_k}(-1)^{|\sigma|}(\tau',\sigma)\left(\sum_{p=1}^{r}\sum_{i=2}^{k+1}(-1)^{i-1}\Big(\mathrm{Cg}_p^{i-1}(\Gamma)\Big)\right)=$$

$$-\sum_{j=1}^{m}\sum_{\tau'\in S_{m-1}}\sum_{\sigma\in S_k}(-1)^{|\sigma|}(\tau_{m,j}\tau',\sigma)\left(\sum_{p=1}^{r}\sum_{i=1}^{k}(-1)^{i-1}\Big(\mathrm{Cg}_p^{i}(\Gamma)\Big)\right)=$$

$$-\sum_{\tau\in S_m}\sum_{\sigma\in S_k}(-1)^{|\sigma|}(\tau,\sigma)\Big(\sum_{p=1}^{r}\sum_{i=1}^{k}(-1)^{i-1}\mathrm{Cg}_p^{i}(\Gamma)\Big)\,.$$

Combining this observation with equation (B.24), we conclude that

$$\mathfrak{d}\mathfrak{d}^*(\gamma)=r\,\gamma \tag{B.27}$$

$$-\sum_{\tau\in S_m}\sum_{\sigma\in S_k}(-1)^{|\sigma|}(\tau,\sigma)\Big(\sum_{p=1}^{r}\sum_{i=1}^{k}(-1)^{i-1}\mathrm{Cg}_p^{i}(\Gamma)\Big)$$

Let us now find a simpler expression for $\mathfrak{d}^*\mathfrak{d}(\gamma)$.

For this purpose we denote by $\omega$ the vector $\mathfrak{d}(\gamma)$ (B.15). By discarding in $\omega$ all graphs for which black vertex with label 1 is not a pike we get the expression

$$\omega'=k\sum_{\tau\in S_{\{2,3,\dots,m+1\}}}\sum_{\sigma'\in S_{k-1}}(-1)^{|\sigma'|}(\tau,\sigma')\left(\sum_{i=1}^{k}(-1)^{i-1}(\tau_{m+1,1},\mathrm{id})(\Gamma\circ_{i,\mathfrak{o}}\Gamma_0^{\mathrm{br}})\right) \tag{B.28}$$

$$+k\sum_{\tau\in S_{\{2,3,\dots,m+1\}}}\sum_{\sigma'\in S_{k-1}}(-1)^{|\sigma'|}(\tau,\sigma')\left(\sum_{p=1}^{r}\sum_{i=1}^{k}(-1)^{i-1}(\varsigma_{m+1,p},\mathrm{id})(\Gamma\circ_{i,\mathfrak{o}}\Gamma_0^{\mathrm{br}})\right)\,.$$

Next, replacing the black vertices with label 1 in each graph in $\omega'$ by a white vertex with label 1 and shifting the labels of the remaining vertices correspondingly, we get

$$\omega''=k\sum_{\tau\in S_m}\sum_{\sigma'\in S_{\{2,3,\dots,k\}}}(-1)^{|\sigma'|}(\tau,\sigma')\left(\sum_{i=1}^{k}(-1)^{i-1}(\mathrm{id},\varsigma_{k,i})(\Gamma)\right) \tag{B.29}$$

$$+k\sum_{\tau\in S_m}\sum_{\sigma'\in S_{\{2,3,\dots,k\}}}(-1)^{|\sigma'|}(\tau,\sigma')\left(\sum_{p=1}^{r}\sum_{i=1}^{k}(-1)^{i-1}\mathrm{Cg}_p^{i}(\Gamma)\right)\,.$$

Thus

$$\mathfrak{d}^*\mathfrak{d}(\gamma)=\sum_{j=1}^{k}\frac{(-1)^{j-1}}{k}(\mathrm{id},\sigma_{k,j})(\omega'')= \tag{B.30}$$

$$\sum_{\tau\in S_m}\sum_{\sigma\in S_k}(-1)^{|\sigma|}(\tau,\sigma)\,\left(\sum_{i=1}^{k}(-1)^{i-1}(\mathrm{id},\varsigma_{k,i})(\Gamma)\right)$$

$$+\sum_{\tau\in S_m}\sum_{\sigma\in S_k}(-1)^{|\sigma|}(\tau,\sigma)\,\left(\sum_{p=1}^{r}\sum_{i=1}^{k}(-1)^{i-1}\mathrm{Cg}^i_p(\Gamma)\right)=$$

$$k\,\gamma+\sum_{\tau\in S_m}\sum_{\sigma\in S_k}(-1)^{|\sigma|}(\tau,\sigma)\,\left(\sum_{p=1}^{r}\sum_{i=1}^{k}(-1)^{i-1}\mathrm{Cg}^i_p(\Gamma)\right)$$

Combining (B.27) with (B.30) we immediately deduce equation (B.11).
Lemma B.3 is proved. □

**Remark B.4** The cochain complex $\mathsf{Hg}$ (B.1) with the differential $\mathfrak{d}$ (B.5) is very similar to Koszul complex for the exterior algebra. However, the author could not find an elegant way to reduce $\mathsf{Hg}$ to this well known complex.

We have the following corollary.

**Corollary B.1** *Let $\gamma$ be a vector in*

$$\mathbf{s}^{2m-2+k}\left(\mathsf{KGra}(m,k)^{\mathfrak{o}}\right)^{S_m}$$

*satisfying Properties A.1, A.2. If $k\ge 1$ and $\gamma$ is $\mathfrak{d}$-closed then there exists*

$$\widetilde{\gamma}\in\mathbf{s}^{2(m-1)-2+k+1}\left(\mathsf{KGra}(m-1,k+1)^{\mathfrak{o}}\right)^{S_{m-1}}$$

*which satisfies Properties A.1, A.2 and such that*

$$\gamma=\mathfrak{d}(\,\widetilde{\gamma}\,)\,. \tag{B.31}$$

Proof. Since $\gamma$ is $\mathfrak{d}$-closed, equation (B.11) implies that

$$\mathfrak{d}\mathfrak{d}^*(\gamma)=k\gamma+\sum_{r\ge1}r\gamma_r\,, \tag{B.32}$$

where $\gamma_r$ is the linear combination in $\mathsf{Hg}$ which is obtained from $\gamma$ by retaining the graphs with exactly $r$ pikes.

Since each graph in the image of $\mathfrak{d}$ has at least one pike, equation (B.32) implies that each graph in the linear combination $\gamma$ has at least one pike. Hence,

$$\gamma=\sum_{r\ge1}\gamma_r \tag{B.33}$$

and (B.32) can be rewritten as

$$\mathfrak{d}\mathfrak{d}^*(\gamma)=\sum_{r\ge1}(k+r)\gamma_r\,. \tag{B.34}$$

Thus, setting

$$\widetilde{\gamma}=\sum_{r\ge1}\frac{1}{k+r}\mathfrak{d}^*(\gamma_r) \tag{B.35}$$

we get the desired identity

$$\gamma=\mathfrak{d}(\,\widetilde{\gamma}\,)\,.$$

□

## C Maurer-Cartan (MC) elements of filtered Lie algebras

Let $\mathcal{L}$ be a Lie algebra in the category $\mathsf{Ch}_{\mathbb{K}}$ of unbounded cochain complexes of $\mathbb{K}$-vector spaces. Let us assume that $\mathcal{L}$ is equipped with a descending filtration

$$\cdots \supset \mathcal{F}_{-1}\mathcal{L} \supset \mathcal{F}_0\mathcal{L} \supset \mathcal{F}_1\mathcal{L} \supset \mathcal{F}_2\mathcal{L} \supset \mathcal{F}_3\mathcal{L} \supset \dots \tag{C.1}$$

which is compatible with the Lie bracket, bounded from the left and such that $\mathcal{L}$ is complete with respect to this filtration. Namely, there exists an integer $m$ such that

$$\mathcal{L} = \mathcal{F}_m\mathcal{L} = \mathcal{F}_{m-1}\mathcal{L} = \mathcal{F}_{m-2}\mathcal{L} = \dots\,, \tag{C.2}$$

and

$$\mathcal{L} = \lim_k \mathcal{L} \big/ \mathcal{F}_k\mathcal{L}\,. \tag{C.3}$$

We call such Lie algebras *filtered*.

Condition (C.3) guarantees that the subalgebra $\mathcal{F}_1\mathcal{L}^0$ of degree zero elements in $\mathcal{F}_1\mathcal{L}$ is a pro-nilpotent Lie algebra (in the category of $\mathbb{K}$-vector spaces). Hence, $\mathcal{F}_1\mathcal{L}^0$ can exponentiated to a pro-unipotent group which we denote by

$$\exp(\mathcal{F}_1\mathcal{L}^0)\,. \tag{C.4}$$

We recall that a *MC element* (Maurer-Cartan element) of $\mathcal{L}$ is a degree 1 vector $\alpha \in \mathcal{L}$ satisfying the equation

$$\partial\alpha + \frac{1}{2}[\alpha, \alpha] = 0\,, \tag{C.5}$$

where $\partial$ denotes the differential on $\mathcal{L}$.

For a vector $\xi \in \mathcal{F}_1\mathcal{L}^0$ and a MC element $\alpha$ we consider the new degree 1 vector $\widetilde{\alpha} \in \mathcal{L}$ which is given by the formula

$$\widetilde{\alpha} = \exp(\mathrm{ad}_\xi)\,\alpha - \frac{\exp(\mathrm{ad}_\xi) - 1}{\mathrm{ad}_\xi}\partial\xi\,, \tag{C.6}$$

where the expressions

$$\exp(\mathrm{ad}_\xi) \qquad \text{and} \qquad \frac{\exp(\mathrm{ad}_\xi) - 1}{\mathrm{ad}_\xi}$$

are defined in the obvious way using the Taylor expansions of the functions

$$e^x \qquad \text{and} \qquad \frac{e^x - 1}{x}$$

around the point $x = 0$, respectively.

Condition (C.3) guarantees that the right hand side of equation (C.6) makes sense.

It is known (see, e.g. [3, Appendix B] or [17]) that, for every MC element $\alpha$ and for every degree zero vector $\xi \in \mathcal{F}_1\mathcal{L}$, the vector $\widetilde{\alpha}$ in (C.6) is also a MC element. Furthermore, formula (C.6) defines an action of the group (C.4) on the set of MC elements of $\mathcal{L}$.

The transformation groupoid corresponding to this action is called the *Deligne groupoid* of the Lie algebra $\mathcal{L}$. This groupoid and its higher versions were studied extensively by E. Getzler in [13] and [14].

**Example C.1** Let $\mathcal{C}$ (resp. $\mathcal{O}$) be a $\Xi$-colored pseudo-cooperad (resp. $\Xi$-colored pseudo-operad) in $\mathsf{Ch}_{\mathbb{K}}$. The convolution Lie algebra $\mathrm{Conv}(\mathcal{C}, \mathcal{O})$ described in Section 2.3 gives us an example of a filtered Lie algebra. Thus it makes sense to talk about the Deligne groupoid of $\mathrm{Conv}(\mathcal{C}, \mathcal{O})$.

### C.1 Differential equations on the Lie algebra $\mathcal{L}\{t\}$

Let $\mathcal{L}$ be a filtered Lie algebra (in $\mathsf{Ch}_{\mathbb{K}}$) such that

$$\mathcal{L} = \mathcal{F}_0 \mathcal{L} \,. \tag{C.7}$$

Using $\mathcal{L}$ we form another Lie algebra $\mathcal{L}\{t\}$ . A vector in $\mathcal{L}\{t\}$ is a formal Taylor power series in an auxiliary (degree zero) variable $t$

$$v = \sum_{k=0}^{\infty} v_k \, t^k \in \mathcal{L}[[t]] \tag{C.8}$$

satisfying the condition

$$\exists \;\; m_0 \ge 0 \quad \text{such that} \quad v_k \in \mathcal{F}_{m_0+k} \mathcal{L} \,. \tag{C.9}$$

The Lie algebra $\mathcal{L}$ is a subalgebra of $\mathcal{L}\{t\}$ formed by the series (C.8) with

$$v_k = 0 \qquad \forall \;\; k \ge 1 \,.$$

Since $\mathcal{L}$ is complete with respect to the filtration $\mathcal{F}_\bullet$, condition (C.9) guarantees that the assignment

$$v \mapsto v \Big|_{t=1} \tag{C.10}$$

defines a Lie algebra homomorphism from $\mathcal{L}\{t\}$ to $\mathcal{L}$ .

The Lie algebra $\mathcal{L}\{t\}$ carries two descending filtrations. First, it carries the descending filtration $\mathcal{F}_\bullet$ which comes from $\mathcal{L}$:

$$\mathcal{F}_m \mathcal{L}\{t\} = \Big\{ \; \sum_{k=0}^{\infty} v_k \, t^k \,, \quad v_k \in \mathcal{F}_{m+k} \mathcal{L} \; \Big\} . \tag{C.11}$$

Second, $\mathcal{L}\{t\}$ carries the $t$-adic filtration from $\mathcal{L}[[t]]$:

$$\mathcal{L}\{t\} \supset t\mathcal{L}\{t\} \supset t^2\mathcal{L}\{t\} \supset t^3\mathcal{L}\{t\} \supset \dots \tag{C.12}$$

We claim that

**Claim C.1** *For every $m$ the Lie subalgebra $\mathcal{F}_m\mathcal{L}\{t\}$ is complete with respect to the filtration*

$$\mathcal{F}_m\mathcal{L}\{t\} \supset t\mathcal{L}\{t\} \cap \mathcal{F}_m\mathcal{L}\{t\} \supset t^2\mathcal{L}\{t\} \cap \mathcal{F}_m\mathcal{L}\{t\} \supset t^3\mathcal{L}\{t\} \cap \mathcal{F}_m\mathcal{L}\{t\} \supset \dots \tag{C.13}$$

Proof. We need to show that any infinite series of the form

$$X = \sum_{p=0}^{\infty} X_p \,, \quad \text{with} \quad X_p \in t^p \mathcal{L}\{t\} \cap \mathcal{F}_m \mathcal{L}\{t\} \tag{C.14}$$

belongs to $\mathcal{F}_m\mathcal{L}\{t\}$ .

Indeed, we have

$$X_p = \sum_{k=p}^{\infty} v_k^p \, t^k$$

for some vectors $v_k^p \in \mathcal{F}_{m+k}\mathcal{L}$ .

Then

$$\sum_{p=0}^{\infty} X_p = \sum_{k=0}^{\infty} t^k \left( \sum_{p=0}^{k} v_k^p \right) \tag{C.15}$$

Since each $v_k^p$ belongs to $\mathcal{F}_{m+k}\mathcal{L}$,

$$\sum_{p=0}^{k} v_k^p \in \mathcal{F}_{m+k}\mathcal{L} \qquad \forall \;\; k \geq 0 \, .$$

Thus the series (C.15) belongs to $\mathcal{F}_m\mathcal{L}\{t\}$ .

Claim C.1 is proved. □

We will need the following proposition

**Proposition C.1** *For every $\alpha \in \mathcal{L}^1$ and $\xi(t) \in \mathcal{F}_1\mathcal{L}^0\{t\}$ the equation*

$$\frac{d}{dt}\alpha(t) = -\partial\xi(t) + [\xi(t), \alpha(t)] \tag{C.16}$$

*with initial condition*

$$\alpha(t)\Big|_{t=0} = \alpha \tag{C.17}$$

*has a unique solution in $\mathcal{L}\{t\}$ . In addition, if $\alpha$ satisfies the Maurer-Cartan equation*

$$\partial\alpha + \frac{1}{2}[\alpha, \alpha] = 0,$$

*then so does $\alpha(t)$ .*

Proof. Let us set up the following iterative procedure in $r \geq 0$

$$\alpha^{(0)}(t) = \alpha$$

and

$$\alpha^{(r)}(t) = \alpha - \int_0^t \partial\xi(t_1)\, dt_1 + \int_0^t [\xi(t_1), \alpha^{(r-1)}(t_1)]\, dt_1 \, . \tag{C.18}$$

Using the inclusion $\xi(t) \in \mathcal{F}_1\mathcal{L}^0\{t\}$, it is not hard to see that $\alpha^{(r)}(t) \in \mathcal{L}\{t\}$ for every $r$. Moreover, Claim C.1 (for $m = 0$) implies that the sequence $\alpha^{(r)}(t)$ converges to an element $\alpha(t) \in \mathcal{L}\{t\}$ .

Since $\alpha(t)$ is constructed via iterative procedure (C.18), $\alpha(t)$ satisfies the integral equation

$$\alpha(t) = \alpha - \int_0^t \partial\xi(t_1)\, dt_1 + \int_0^t [\xi(t_1), \alpha(t_1)]\, dt_1 \tag{C.19}$$

and hence differential equation (C.16) with initial condition (C.17).

To prove the uniqueness, let us assume that $\widetilde{\alpha}(t)$ is another solution of (C.16) with the initial condition (C.17). Then the difference:

$$\psi(t) = \widetilde{\alpha}(t) - \alpha(t)$$

satisfies the differential equation

$$\frac{d}{dt}\psi(t) = [\xi(t), \psi(t)] \tag{C.20}$$

with the initial condition

$$\psi(t)\Big|_{t=0} = 0\,. \tag{C.21}$$

Using (C.20) and (C.21) we conclude that

$$\psi(t) = \int_0^t [\xi(t_1), \psi(t_1)]\, dt_1 \tag{C.22}$$

The initial condition (C.21) implies that

$$\psi(t) \in t^p \mathcal{L}\{t\} \tag{C.23}$$

for some $p \ge 1$.

However, inclusion (C.23) and the integral equation (C.22) imply that $\psi(t) \in t^{p+1}\mathcal{L}\{t\}$. Thus

$$\psi(t) \in \bigcap_{p\ge 1} t^p \mathcal{L}\{t\}$$

and hence $\psi(t) = 0$.

Thus we proved the first statement of Proposition C.1

To prove the second statement we consider the following element

$$\Psi(t) = d\alpha(t) + \frac{1}{2}[\alpha(t), \alpha(t)] \in \mathcal{L}^1\{t\}\,. \tag{C.24}$$

Taking a derivative in $t$ and using (C.16), we get

$$\frac{d}{dt}(d\alpha(t) + \frac{1}{2}[\alpha(t), \alpha(t)]) = [\xi(t), d\alpha(t) + \frac{1}{2}[\alpha(t), \alpha(t)]]\,.$$

In other words, the element $\Psi(t)$ satisfies the differential equation

$$\frac{d}{dt}\Psi(t) = [\xi(t), \Psi(t)]\,. \tag{C.25}$$

Since $\alpha$ satisfies the MC equation, we conclude that

$$\Psi(t)\Big|_{t=0} = 0\,. \tag{C.26}$$

Combining (C.25) and (C.26) we deduce the integral equation for $\Psi(t)$, that is

$$\Psi(t) = \int_0^t [\xi(t_1), \Psi(t_1)]\, dt_1\,. \tag{C.27}$$

On the other hand, (C.27) implies that

$$\Psi(t) \in \bigcap_{p\ge 1} t^p \mathcal{L}\{t\}$$

and hence $\Psi(t) = 0$.

Proposition C.1 is proved. $\square$

Proposition C.1 implies that using an element $\xi(t) \in \mathcal{F}_1 \mathcal{L}^0\{t\}$ and a MC element $\alpha \in \mathcal{L}$ we can produce another MC element $\alpha'$ by solving equation (C.16) with initial condition (C.17) and setting

$$\alpha' = \alpha(t)\Big|_{t=1} .$$

Theorem C.1 below states that these MC elements are isomorphic. However, to prove this theorem we will need the following technical statement:

**Lemma C.2** *If $\alpha$ is a MC element of $\mathcal{L}$, $\xi(t) \in \mathcal{F}_1 \mathcal{L}^0\{t\}$, and $\alpha(t)$ is the unique solution of (C.16) with initial condition (C.17), then for every $\eta \in \mathcal{F}_{k+1}\mathcal{L}^0$ and every nonnegative integer $k$, the element*

$$\beta(t) = \exp\left(\frac{t^{k+1}}{k+1} \mathrm{ad}_\eta\right) \alpha(t) - \frac{\exp\left(\frac{t^{k+1}}{k+1}\mathrm{ad}_\eta\right) - 1}{\mathrm{ad}_\eta}\, \partial\eta \qquad \text{(C.28)}$$

*satisfies the differential equation*

$$\frac{d}{dt}\beta(t) = [\widetilde{\xi}, \beta(t)] - \partial\widetilde{\xi}\,, \qquad \text{(C.29)}$$

*where*

$$\widetilde{\xi} = t^k \eta + \exp\left(\frac{t^{k+1}}{k+1}\mathrm{ad}_\eta\right) \xi\,. \qquad \text{(C.30)}$$

Proof. First, we remark that, the infinite series in (C.28) and (C.30) are well defined due to Claim C.1.

Second, we recall that the following identity

$$e^X \partial(e^{-X}) = -\frac{e^{\mathrm{ad}_X} - 1}{\mathrm{ad}_X}\, \partial X \qquad \text{(C.31)}$$

holds in any differential graded associative algebra provided the infinite series in both sides make sense.

Applying this identity to the element

$$X = \frac{t^{k+1}}{k+1}\mathrm{ad}_\eta$$

in the algebra of endomorphisms of $\mathcal{L}^0\{t\}$ we deduce that

$$U_\eta\big(\partial U_\eta^{-1}(v)\big) - v = -\left[\ \frac{\exp\left(\frac{t^{k+1}}{k+1}\mathrm{ad}_\eta\right) - 1}{\mathrm{ad}_\eta}\, \partial\eta\ ,\ v\ \right], \qquad \text{(C.32)}$$

for all $v \in \mathcal{L}^0\{t\}$, where

$$U_\eta = \exp\left(\frac{t^{k+1}}{k+1}\mathrm{ad}_\eta\right) . \qquad \text{(C.33)}$$

Using (C.32) for
$$v = U_\eta(t^k \eta + \xi(t))$$
we get the identity
$$U_\eta\big(\partial(t^k\eta + \xi(t))\big) - U_\eta(t^k\eta + \xi(t)) = \left[\, U_\eta(t^k\eta + \xi(t))\ ,\ \frac{\exp\left(\frac{t^{k+1}}{k+1}\mathrm{ad}_\eta\right) - 1}{\mathrm{ad}_\eta}\ \partial\eta \,\right]. \quad \text{(C.34)}$$

On the other hand, since $[\eta, \eta] = 0$, we have $U_\eta(t^k\eta) = t^k\eta$. Hence, (C.34) gives us
$$U_\eta\big(\partial(t^k\eta+\xi(t))\big) - \big(t^k\eta + U_\eta(\xi(t))\big) = \left[\, \big(t^k\eta + U_\eta(\xi(t))\big)\ ,\ \frac{\exp\left(\frac{t^{k+1}}{k+1}\mathrm{ad}_\eta\right) - 1}{\mathrm{ad}_\eta}\ \partial\eta \,\right]. \quad \text{(C.35)}$$

Using (C.16) together with identity (C.35) it is not hard to prove (C.29) by direct computation.

Lemma C.2 is proved.

□

Let us now prove a statement which is used in Section 6.2.

**Theorem C.1** *Let $\mathfrak{g}$ be a Lie subalgebra of $\mathcal{L}^0$. If $\alpha$ is a MC element of $\mathcal{L}$, $\xi(t) \in \mathcal{F}_1\mathfrak{g}\{t\}$, and $\alpha(t)$ is the unique solution of (C.16) with initial condition (C.17), then the MC elements*
$$\alpha \qquad \textit{and} \qquad \alpha(t)\Big|_{t=1}$$
*are connected by the action* (C.6) *of the pro-unipotent group*
$$\exp(\mathcal{F}_1\mathfrak{g})\,.$$

Proof. The statement of this theorem is very similar to [3, Proposition B.7]. Unfortunately, theorem C.1 is not a Corollary of [3, Proposition B.7]. So we give a separate proof.

In general,
$$\xi(t) = \xi_0 + \xi_1 t + \xi_2 t^2 + \dots,$$
where $\xi_k \in \mathcal{F}_{1+k}\mathfrak{g}$.

Applying Lemma C.2 with $k = 0$ to $\eta = -\xi_0$ we get the MC element
$$\alpha_1(t) = \exp\left(-t\,\mathrm{ad}_{\xi_0}\right)\alpha(t) - \frac{\exp\left(-t\,\mathrm{ad}_{\xi_0}\right) - 1}{\mathrm{ad}_{\xi_0}}\ \partial\xi_0 \qquad \text{(C.36)}$$
which solves the initial value problem
$$\frac{d}{dt}\alpha_1(t) = [\xi^{(1)}(t), \alpha_1(t)] - \partial\xi^{(1)}(t)\,, \qquad \alpha_1(t)\Big|_{t=0} = \alpha \qquad \text{(C.37)}$$
with
$$\xi^{(1)}(t) = \exp\left(-t\,\mathrm{ad}_{\xi_0}\right)\xi(t) - \xi_0\,. \qquad \text{(C.38)}$$

Next, we observe that $\xi^{(1)} \in t\mathfrak{g}\{t\} \cap \mathcal{F}_1\mathfrak{g}\{t\}$. In other words,
$$\xi^{(1)}(t) = \xi^{(1)}_1 t + \xi^{(1)}_2 t^2 + \xi^{(1)}_3 t^3 + \dots,$$

where $\xi_k^{(1)} \in \mathcal{F}_{1+k}\mathfrak{g}$.

Applying Lemma C.2 with $k = 1$, $\xi(t)$ replaced by $\xi^{(1)}(t)$ to $\eta = -\xi_1^{(1)}$ we get yet another MC element

$$\alpha_2(t) = \exp\left(-\frac{t^2}{2}\,\mathrm{ad}_{\xi_1^{(1)}}\right)\alpha_1(t) - \frac{\exp\left(-\frac{t^2}{2}\,\mathrm{ad}_{\xi_1^{(1)}}\right) - 1}{\mathrm{ad}_{\xi_1^{(1)}}}\,\partial\xi_1^{(1)} \tag{C.39}$$

which solves the initial value problem

$$\frac{d}{dt}\alpha_2(t) = [\xi^{(2)}(t), \alpha_2(t)] - \partial\xi^{(2)}(t)\,, \qquad \alpha_2(t)\Big|_{t=0} = \alpha \tag{C.40}$$

with

$$\xi^{(2)}(t) = \exp\left(-\frac{t^2}{2}\,\mathrm{ad}_{\xi_1^{(1)}}\right)\xi^{(1)}(t) - \xi_1^{(1)}t\,. \tag{C.41}$$

It is easy to see that $\xi^{(2)}(t)$ lies in the deeper filtration subalgebra $t^2\mathfrak{g}\{t\} \cap \mathcal{F}_1\mathfrak{g}\{t\}$.

Repeating these steps over and over again we get an iterative procedure. The output of this procedure is a pair of infinite sequences:

$$\alpha(t),\ \alpha_1(t),\ \alpha_2(t),\ \alpha_3(t),\ \ldots,\ \alpha_k(t),\ \ldots \tag{C.42}$$

$$\xi(t),\ \xi^{(1)}(t),\ \xi^{(2)}(t),\ \xi^{(3)}(t),\ \ldots,\ \xi^{(k)}(t), \ldots \tag{C.43}$$

where $\alpha_k(t)$'s are MC elements of $\mathcal{L}\{t\}$ and

$$\xi^{(k)}(t) = \xi_k^{(k)}t^k + \xi_{k+1}^{(k)}t^{k+1} + \xi_k^{(k+2)}t^{k+2} + \cdots \in t^k\mathfrak{g}\{t\} \cap \mathcal{F}_1\mathfrak{g}\{t\}\,.$$

Furthermore, the pair $(\alpha_k(t), \xi^{(k)}(t))$ is obtained from $(\alpha_{k-1}(t), \xi^{(k-1)}(t))$ via

$$\alpha_k(t) = \exp\left(-\frac{t^k}{k}\,\mathrm{ad}_{\xi_{k-1}^{(k-1)}}\right)\alpha_{k-1}(t) - \frac{\exp\left(-\frac{t^k}{k}\,\mathrm{ad}_{\xi_{k-1}^{(k-1)}}\right) - 1}{\mathrm{ad}_{\xi_{k-1}^{(k-1)}}}\,\partial\xi_{k-1}^{(k-1)} \tag{C.44}$$

$$\xi^{(k)}(t) = \exp\left(-\frac{t^k}{k}\,\mathrm{ad}_{\xi_{k-1}^{(k-1)}}\right)\xi^{(k-1)}(t) - \xi_{k-1}^{(k-1)}t^{k-1} \tag{C.45}$$

and $\alpha(t)$ solves the initial value problem

$$\frac{d}{dt}\alpha_k(t) = [\xi^{(k)}(t), \alpha_k(t)] - \partial\xi^{(k)}(t)\,, \qquad \alpha_k(t)\Big|_{t=0} = \alpha \tag{C.46}$$

It is easy to see that the sequence $\alpha_k(t)$ converges in $\mathcal{L}\{t\}$ to $\alpha$ and, moreover, if

$$\alpha' = \alpha(t)\Big|_{t=1}$$

then

$$\alpha = \ldots \exp\left(-\frac{1}{3}\xi_2^{(2)}\right)\exp\left(-\frac{1}{2}\xi_1^{(1)}\right)\exp(-\xi_0)\alpha'\,.$$

Theorem C.1 is proved.

$\square$

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Department of Mathematics, Temple University,
Wachman Hall Rm. 638
1805 N. Broad St.,
Philadelphia PA, 19122 USA
*E-mail address:* **vald@temple.edu**